A Pure Power Product Version of the Hilbert Nullstellensatz

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I. Background

Let k be a field and $R = k[x_0, \dots, x_n]$. Then what one might call the *radical version* of Hilbert's Nullstellensatz states that, for any homogeneous ideal $\mathfrak{A} = (f_1, \dots, f_m)$ with radical $\mathfrak{R}$, some power of $\mathfrak{R}$ lies in $\mathfrak{A}$:

$$\mathfrak{R}^e \subset \mathfrak{A}.$$

From now on, let us denote by e the minimum such exponent for this $\mathfrak{A}$.

Rabinowitsch [Ra] showed that this formulation is equivalent to the following (apparently weaker) assertion, which has been called the *Bezout version* of the Nullstellensatz: If $g_1, \dots, g_m$ in $S = k[x_1, \dots, x_n]$ have no common zeros (say, in an algebraic closure of k), then there exist $a_1, \dots, a_m$ in S such that

$$1 = a_1 g_1 + \dots + a_m g_m.$$

Denote by a the minimal value of $\max \deg a_i$ of all choices of a_i in S satisfying this identity.

If $d_i = \deg f_i = \deg g_i > 0$ $(i = 1, \dots, m)$, then general upper bounds for e and a are intimately related, and here we regard them as equivalent. Nearly optimal bounds for a and e were achieved almost a decade ago. To discuss these bounds, for the remainder of the paper let us order the degrees so that

$$D = d_2 \ge d_3 \ge \dots \ge d_m \ge d_1.$$

The classical work of Hermann [He] was taken up again by Masser and Wüstholz [MW] to establish the first effective version of the Nullstellensatz. They showed that, in the Bezout form,

$$a \le 2(2D)^{2n-1}.$$

Masser and Philippon gave a family of examples, which was refined a bit in [B2] to show that, in certain cases,

$$a = D^n - D;$$

correspondingly, $e \ge D^n$. Another family of examples was devised by Kollár [Ko].

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