# The Total Mean Curvature of Submanifolds in a Euclidean Space

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## 1. Introduction

Let $M^n$ be an $n$-dimensional submanifold in a Euclidean space $E^{n+p}$ of dimension $n+p$. Denote by $R$ the normalized scalar curvature and by $H$ the mean curvature of $M^n$.

Ōtsuki [O] introduced a kind of curvatures, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ for $M^2$ in $E^{2+p}$, and showed that they can be used to study the geometry of surfaces in higher-dimensional Euclidean space. Shiohama [S] proved that a complete oriented surface in $E^{2+p}$ with $\lambda_\alpha = 0$ $(1 \leq \alpha \leq p)$ is a cylinder. Chen [C1] classified compact oriented surfaces in $E^{2+p}$ with $\lambda_p \geq 0$.

In higher-dimensional cases, Chen [C3] introduced the notion of $\alpha$th scalar curvatures, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ for $M^n$ in $E^{n+p}$, and found a relationship between the $\alpha$th scalar curvatures and the scalar curvature. When $n=2$, it reduces to that introduced by Ōtsuki [O]. Chen [C3] also proved that a closed submanifold $M^n$ $(n \geq 3)$ in $E^{n+p}$ with $\int_{M^n} (\lambda_1)^{n/2}\, dV = c_n$ and $\lambda_\alpha = 0$ $(2 \leq \alpha \leq p)$ is an $n$-sphere, where $c_n$ is the volume of the unit $n$-sphere and $dV$ denotes the volume element of $M^n$.

In this paper, we give a further description of the behavior of the $\alpha$th scalar curvatures and obtain some applications of them. In Section 2, we first prove that $\lambda_\alpha \leq 0$ $(2 \leq \alpha \leq p)$ for any submanifold $M^n$ in $E^{n+p}$. Then we prove an inequality involving the integral of $\lambda_1$ for closed $M^n$ in $E^{n+p}$ with $R \geq 0$.

Suppose that $M^n$ is closed in $E^{n+p}$. The total mean curvature of $M^n$ is defined to be the integral $\int_{M^n} H^n\, dV$. An interesting and outstanding problem is to find the best possible lower bound of this integral in terms of the geometric or topologic invariants of $M^n$. A special case of this problem is the famous Willmore's conjecture. There have been many results obtained on this problem. In Section 3 we give an estimate of the total mean curvature for closed submanifolds in $E^{n+p}$ with $R \geq 0$. The main result of this paper is the following theorem.

Theorem 3.1. *Let $M^n$ be a closed submanifold in $E^{n+p}$ with $R \geq 0$. Then*

$$\int_{M^n} H^n\, dV \geq 2\kappa_n c_{n-1} + \left\{1 - 2\kappa_n\left(\frac{c_{n-1}}{c_n}\right)\right\} \int_{M^n} R^{n/2}\, dV,$$

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