Standard Forms of 3-Braid 2-Knots and their Alexander Polynomials

Seiichi Kamada

By a *surface link* we mean a closed oriented locally flat surface F in 4-space $\mathbf{R}^4$. It is called a *closed 2-dimensional braid* of degree m if it is contained in a tubular neighborhood $N(S^2) \cong D^2 \times S^2$ of a standard 2-sphere S^2 in $\mathbf{R}^4$ such that the restriction to F of the projection $D^2 \times S^2 \to S^2$ is a degree-m simple branched covering map from F to S^2. Viro [V; cf. K2; CS] proved that every surface link is ambient isotopic to a closed 2-dimensional braid of degree m for some m. The *braid index* of F, denoted by $\mathrm{Braid}(F)$, is the minimum degree among all closed 2-dimensional braids ambient isotopic to F.

By definition, $\mathrm{Braid}(F) = 1$ if and only if F is an unknotted 2-sphere (i.e., ambient isotopic to the standard 2-sphere in $\mathbf{R}^4$). It is easily seen that $\mathrm{Braid}(F) = 2$ if and only if F is an unknotted surface link in $\mathbf{R}^4$ that is a connected surface with nonnegative genus or a pair of 2-spheres; cf. [K1]. (A surface link is *unknotted* if it bounds mutually disjoint locally flat 3-balls or handlebodies in $\mathbf{R}^4$. This condition is equivalent to its being isotoped into a hyperplane of $\mathbf{R}^4$; see [HK].) In particular, there exist no 2-knots of braid index 2.

Our interest is 3-braid 2-knots, that is, 2-spheres in $\mathbf{R}^4$ of braid index 3. The spun 2-knot of a $(2, q)$-type torus knot is a 3-braid 2-knot unless $q = \pm 1$. Of course, there exist infinitely many 3-braid 2-knots which are not spun 2-knots.

Few results on 3-braid 2-knots are known. For example, all 3-braid 2-knots—and all surface links of braid index 3 or less—are ribbon [K1]. (A surface link is said to be *ribbon* if it is obtained from a split union of unknotted 2-spheres by surgery along some 1-handles attached to them.) Thus the 2-twist spun 2-knot of a trefoil knot is not a 3-braid 2-knot.

The purpose of this paper is to prove that a 3-braid 2-knot can always be deformed into a certain kind of configuration, called a *standard form* (Section 1). In Section 2 we investigate Alexander polynomials of 3-braid 2-knots by use of standard forms. Our main theorem (Theorem 2.3) regards a strong relationship between standard forms and the spans of the Alexander polynomials. (The *span* means the maximal degree minus the minimal.) Using it, we obtain some results on Alexander polynomials of 3-braid 2-knots; for instance, nontriviality of them. Standard forms (and Alexander polynomials) are quite useful for distinguishing

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