Polynomial Hulls with Disk Fibers over the Ball in $\mathbf{C}^2$

MARSHALL A. WHITTLESEY

Let Y be a compact set in $\mathbf{C}^n$. We denote by $\hat{Y}$ the *polynomial* (*convex*) *hull* of Y; that is,

$$\hat{Y} = \{ z \in \mathbf{C}^n \mid |P(z)| \leq \sup_{w \in Y} |P(w)| \text{ for all polynomials } P \text{ on } \mathbf{C}^n \}.$$

Let B_2 denote the open unit ball in $\mathbf{C}^2$. We shall be interested in the polynomial hulls of compact sets Y in $\mathbf{C}^3$, where Y projects onto $S = \partial B_2$.

The analogous problem of describing the polynomial hull of a compact set fibered over the unit circle Γ in $\mathbf{C}$ has been studied in [1; 3; 8; 9; 11]. A major issue in all of these works is to describe the extent of *analytic structure*—that is, when there exist analytic varieties contained in the polynomial hull with boundary contained in Y. One reason for this is that discovering such a variety in $\hat{Y}$ explains why the points on that variety lie in $\hat{Y}$ by virtue of the well-known local maximum modulus principle on analytic varieties. In this work, we shall examine when one can expect a higher degree of analytic structure, that is, when there exist analytic manifolds of dimension 2 in $\hat{Y}$ with boundary in Y. In particular, we shall examine when such an analytic manifold is in fact the graph of an analytic function over B_2.

Let Δ denote the closed unit disk in $\mathbf{C}$. In [11] Wermer showed that, for hulls of sets fibered over the circle, there need never exist analytic structure. However, it was shown by Alexander and Wermer in [1] and by Słodkowski in [8] that if Y has convex fibers Y_λ ($\lambda \in \Gamma$) over the circle then $\hat{Y} \cap \{|\lambda| < 1\}$ is the union of analytic graphs over int Δ of functions f in $H^\infty(\Delta)$ such that $f(\lambda) \in Y_\lambda$ for a.e. $\lambda \in \Gamma$. Furthermore, Alexander and Wermer proved the following.

THEOREM [1, Thm. 2]. *Suppose that α is a continuous complex-valued function on $\Gamma = \{|\lambda| = 1\}$, $\|\alpha\|_\infty \leq 1$. Put $Y = \{ (\lambda, w) \mid |w - \alpha(\lambda)| \leq 1, \lambda \in \Gamma \}$. Assume there is a b with $|b| < 1$ such that $\hat{Y}_b$ contains more than one point. Then there exist functions A, B, C, analytic on $\{|\lambda| < 1\}$ and in H^p ($0 < p < 1$), as well as a $\phi_0 \in H^\infty$ such that*

$$\hat{Y} \cap \{|\lambda| < 1\} = \left\{ (\lambda, w) \;\middle|\; \left| \frac{A(\lambda)(w - \phi_0(\lambda)) + C(\lambda)}{B(\lambda)(w - \phi_0(\lambda)) + C(\lambda)} \right| \leq 1,\ |\lambda| < 1 \right\}.$$

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