On Iteration in Planar Domains

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Introduction

Let $G \subset \hat{\mathbb{C}}$ be a domain in the complex plane, and let $f: G \to G$ be an analytic function mapping G into itself. By f^n we denote the nth iterate $f^n = f \circ \cdots \circ f$ (n times) of f. The behavior of the sequence $(f^n)_n$ as $n \to \infty$ is of great interest and has already been studied in depth for the most important choices of G. If $G = \hat{\mathbb{C}}$, then f is a rational function. Many significant results have been proved during the last years (see e.g. [B2; CG; Mi; S]). In the case $G = \mathbb{C}$, the function f is an entire function; see [Be] for an excellent overview. If G is the unit disk, $G = \mathbb{D}$, then the situation becomes easier, since in this case the family $\{ f^n \mid n \in \mathbb{N}\}$ is normal. Initial results have been discovered by Julia, Wolff, and Valiron (see e.g. [V]); further results have been found by Pommerenke and Baker [P; BaP] and by Cowen [C].

Owing to the uniformization theorem of Koebe and Poincaré, any other domain $G \subset \hat{\mathbb{C}}$ (and even every Riemann surface) is conformally equivalent to the quotient of one of these standard domains and a discrete, fixed-point free subgroup of the automorphism group associated with that domain. Hence, it seems obvious that the behavior of the iterates of an analytic function in an arbitrary domain can be deduced from one of the cases mentioned above. If f possesses a fixed point in G then this statement is true, but if not then the boundary of G becomes significant and the boundary behavior of the universal covering map must be examined. The cases where G is covered by the whole complex plane are represented by the plane $\mathbb{C}$ itself with entire functions being iterated and by the punctured plane $\mathbb{C}^*$, the latter case being reduced to the first one by means of the exponential map that covers $\mathbb{C}^*$ by $\mathbb{C}$ and whose boundary behavior is well known. The only case where $G \subset \hat{\mathbb{C}}$ is covered by $\hat{\mathbb{C}}$ is the case $G = \hat{\mathbb{C}}$, so that this case does not introduce new problems compared with the iteration of rational functions.

From now on, let $G \subset \hat{\mathbb{C}}$ be a planar domain covered by the unit disk $\mathbb{D}$, that is, G has at least three boundary points in $\hat{\mathbb{C}}$. We call such domains *hyperbolic*, since they carry a hyperbolic metric (see below). Note that some authors use the term "hyperbolic" differently: they call a domain G hyperbolic if it possesses a Green's function. In [Hei] the iteration of analytic functions in hyperbolic domains has already been studied, in particular the case of f having a fixed point in G is dealt

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