Spaces and Arcs of Bounded Turning

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1. Introduction

A closed subset X of R^n is of *bounded turning* if there is a fixed number $c \geq 1$ such that any two points a and b of X can be joined by a connected subset A of X with the diameter $d(A)$ of A satisfying

$$d(A) \leq c|a - b|. \tag{1}$$

We will abbreviate bounded turning as BT, and A is c-BT if (1) is true with this particular c. The aim of this paper is to prove the following theorem.

THEOREM 1A. *There is $c_0 = c_0(c, n)$ such that any two points in a c-BT set X of R^n can be joined by a c_0-BT arc J of X.*

Originally, the notion of bounded turning was introduced for arcs or topological circles in R^2. The BT condition characterizes such arcs or circles that are images of the standard interval $[0, 1]$ or of the unit circle S^1 under a quasiconformal homeomorphism of the plane; see [A] and [R].

Hence this notion has an honorable standing in the theory of quasiconformal mappings of the plane. In higher dimensions, the BT property is a necessary condition for an arc or a circle to be the image of the standard interval or circle under a quasiconformal map of R^n, but the condition is no longer sufficient. For instance, the Fox–Artin wild arc in R^3 can be made BT. ([Ma] discusses this in the situation where the Fox–Artin arc is fattened so as to obtain a wild sphere; cf. also [T, Sec. 14 and Sec. 17].) On the other hand, we might more modestly want to map only an interval of the real axis onto an arc of R^n using a map that would have the same properties as the restriction of a quasiconformal map of R^n to the interval. Such maps are called *quasisymmetric maps* (see [TV]). Now the BT property characterizes when such an arc or a circle is a quasisymmetric image of a standard interval or circle [TV, Thm. 4.9].

The question of whether any two points of BT space X of R^n can be joined by a BT arc of X was raised by J. Väisälä. I am indebted to him for this very interesting question—whose answer turned out to be much more complicated than anticipated—and for some critical comments on my earlier attempts to prove the

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