A Rigidity Theorem for Composition Operators on Certain Bergman Spaces

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Let ϕ be an analytic self-map of the open unit disk $\mathbf{D}$ in the complex plane. We consider the composition operator C_ϕ, defined by $C_\phi f = f \circ \phi$, acting on a weighted Bergman space A_G^2. Here $G(r)$, $0 < r < 1$, is a positive continuous function and A_G^2 consists of all f analytic on $\mathbf{D}$ with

$$\|f\|^2 \stackrel{\text{def}}{=} \int_{\mathbf{D}} |f(z)|^2 G(|z|)\, dA < \infty, \tag{1}$$

where dA is area measure on $\mathbf{D}$. We assume that G is non-increasing and that $G(|z|)$ is dA-integrable over $\mathbf{D}$. It is well known that the norm $\|\cdot\|$ defined by (1) makes A_G^2 into a Hilbert space. The purpose of this note is to locate a family of "critical weights" G_* with the property that any A_G^2 defined from a weight G which tends to zero more rapidly than G_* admits only compact and unitary composition operators.

It is known that if $G(r) = (1-r)^\alpha$ with $\alpha \geq 0$ (the standard weights), then every C_ϕ defines a bounded operator on A_G^2. Moreover, C_ϕ is a compact operator on these spaces exactly when ϕ has no finite angular derivative at any point on $\partial\mathbf{D}$. Recall that if ζ lies in the unit circle $\partial\mathbf{D}$, ϕ is said to have a (finite) angular derivative $\phi'(\zeta)$ at ζ if there exists w in $\partial\mathbf{D}$ such that

$$\phi'(\zeta) \stackrel{\text{def}}{=} \lim_{z \to \zeta} \frac{\phi(z) - w}{z - \zeta}$$

exists, where $z \to \zeta$ nontangentially. This happens exactly when the quantity

$$\liminf_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|} \tag{2}$$

is finite, where here $z \to \zeta$ unrestrictedly in $\mathbf{D}$; in this case expression (2) coincides with $|\phi'(\zeta)|$. Let us write $|\phi'(\zeta)|$ for (2) even when the lim inf is infinite. Note that when $\phi'(\zeta)$ exists as a finite limit, the nontangential limit of ϕ at ζ, call it $\phi(\zeta)$, exists and has modulus 1. Thus if the nontangential limit $\phi(\zeta)$ fails to exist, or if it exists but $|\phi(\zeta)| \neq 1$, then $|\phi'(\zeta)| = \infty$. If $\phi(\zeta) = \zeta$ and $\phi'(\zeta)$ exists, it is positive. For any ϕ and ζ, we have $0 < |\phi'(\zeta)| \leq \infty$. Thus compactness of C_ϕ on the standard weight spaces is characterized by: $|\phi'(\zeta)| = \infty$ for all ζ in $\partial\mathbf{D}$ (see [6]).

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