The Best Approximation and Composition with Inner Functions

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0. Introduction

Let $f \in L^p(T)$, $0 < p \le +\infty$, where T is the unit circle in the complex plane C and $\mathcal{P}_n$ the set of complex polynomials whose degree is not greater than n. The best approximation of f by the polynomials of degree n in L^p-metric is defined by

$$E_n(f) = E_n(f)_p = \inf\{\|f-g\|_p : g \in \mathcal{P}_n\}.$$

An application of the Hahn-Banach theorem to the quotient space $L^p/\mathcal{P}_n$, $1 \le p < +\infty$, enables us to get a convenient expression of $E_n(f)_p$ for further application (see the equality (1.1) below). Using (1.1), we prove that if $F \in H^p$ $(1 < p < +\infty)$, w is an inner function, and $w(0) = 0$, then $E_n(F \circ w)_p \ge E_n(F)_p$.

In Sections 2 and 3, we consider applications of this result. In Section 2, we give a characterization of the Lipschitz-Besov space $X = A(\alpha, p, q)$ by means of the best approximation. Using this result, we prove that if $F \circ w \in X$ then $F \in X$, where X is a Lipschitz-Besov space and w is an inner function.

If φ is an inner function and if φ has no zeros in the unit disk then $\varphi = A \circ w$, where w is an inner function and A is the atomic function. In Section 3, combining this fact with results obtained in previous sections, we prove that if φ is an inner function with a nonconstant singular factor and $1 \le p < +\infty$, then there exists a positive constant C such that

$$E_k(\varphi)_p \ge Ck^{-1/2p}.$$

This result is sharp. The case $p = 2$ of this result is due to Newman and Shapiro [6].

Finally, applications to the growth of integral means of inner functions are given. Using the lower bound on the rate at which $E_n(f)_p$ may go to zero, together with the characterization of $A_0(p)$, we prove that if φ is an inner function with nonconstant singular factor and $0 < p < +\infty$ then

$$\lim_{r \to 1_-} \sup (1-r)^{1-1/2p} M_p(r, \varphi') > 0.$$

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