# Regularity of the Dirichlet Problem in Convex Domains in the Plane

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## 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Then $\partial\Omega$ is Lipschitz but generally not $C^1$. Consider the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

The regularity of $u$ is limited by the regularity of $\partial\Omega$. In this paper, it is shown that for sufficiently smooth $f$, $u$ has almost three derivatives in $L^1(\Omega)$. Specifically, we will establish the following theorem concerning the operator $G$ defined by $Gf = u$.

THEOREM 1.1. *If $\Omega \subset \mathbb{R}^2$ is bounded and convex, $0 < \epsilon < 1$, $1 < p < 2/(2-\epsilon)$, and $f \in L^p_{1-\epsilon}(\Omega)$, then for some $C$,*

$$Gf \in L^p_{3-\epsilon}(\Omega) \quad \text{and} \quad \|Gf\|_{L^p_{3-\epsilon}(\Omega)} \le C\|f\|_{L^p_{1-\epsilon}(\Omega)}. \tag{2}$$

The following theorem, stated in [7], summarizes some previous regularity results for problem (1) in bounded convex domains in $\mathbb{R}^n$, $n \ge 2$.

THEOREM 1.2. *If $\Omega \subset \mathbb{R}^n$ is bounded and convex and $n \ge 2$, and either $1 \le \epsilon \le 2$ and $1 < p < 2/(2-\epsilon)$ or $\epsilon = 1$ and $p = 2$, then* (2) *holds. The Dirichlet boundary condition is satisfied in the sense that $Gf \in W^{1,p}_0(\Omega)$.*

The case $\epsilon = 1$ and $p = 2$ follows from integration by parts and is due to Kadlec [12]; an accessible exposition may be found in the third chapter of [9] or [10]. The case $\epsilon = 1$ and $1 < p < 2$ is due to Dahlberg, Verchota, and Wolff (see [7]) and independently to Adolfsson [1]. The proof of the result for $\epsilon = 2$ may be found in [7].

In particular, even if $f \in C^\infty(\bar{\Omega})$, the best one can say about $\nabla u$ and $\nabla^2 u$ is that $\nabla u \in L^\infty(\Omega)$ and $\nabla^2 u \in L^2(\Omega)$. In addition, by considering a family of truncated convex sectors in $\mathbb{R}^2$ with opening angles increasing to $\pi$, it

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