Cohomology of the Symplectic Group Sp(4, Z), Part II: Computations at the Prime 2

ALAN BROWNSTEIN & RONNIE LEE

0. Introduction

In this paper, we give a complete description of the integral cohomology group $H^*(\mathrm{Sp}(4,\mathbf{Z});\mathbf{Z})$ of $\mathrm{Sp}(4,\mathbf{Z})$. In our previous paper [BL], the odd torsion pieces were determined, so here we concentrate on the 2-primary components. These components are closely related to the cohomology of the mapping class group Γ_2^0 of a genus-2 Riemann surface. Using this relationship and the knowledge of $H^*(\Gamma_2^0;\mathbf{Z})$ in the recent work of Benson and Cohen, we complete the project started in [BL].

Let $\mathrm{Sp}(4;\mathbf{Z})$ denote the group of 4-by-4 integral matrices preserving the skew symmetric pairing $\begin{pmatrix} 0 & I \\ -1 & 0 \end{pmatrix}$. This is a well-known arithmetic subgroup of the real symplectic group $\mathrm{Sp}(4;\mathbf{R})$ and has been studied by various authors from the viewpoint of automorphic forms (e.g. see [G1; G2; Ba]). For $\mathrm{Sp}(4;\mathbf{R})$ is a generalization of the special linear group $\mathrm{Sp}(2;\mathbf{R})=\mathrm{SL}(2;\mathbf{R})$, and the theory of automorphic forms on $\mathrm{Sp}(4;\mathbf{R})$ can be regarded as a natural extension of the study of elliptic functions. From a topological viewpoint, this group is of some interest because of its relation with the mapping class groups Γ_2^0 and Γ_0^6. In general, let S_g^n be a surface of genus g with n punctures, and let $\mathrm{diff}^+(S_g^n)$ be the group of orientation-preserving diffeomorphisms of the surface S_g^n which fix the punctures setwise. The mapping class group Γ_g^n is the set of isotopy classes of $\mathrm{diff}^+(S_g^n)$. Since the cohomology $H^*(\Gamma_2^0;\mathbf{Z})$ can be identified with the cohomology $H^*(B\Gamma_2^0;\mathbf{Z})$ of the classifying space $B\Gamma_2^0$, classes in $H^*(\Gamma_2^0;\mathbf{Z})$ can be regarded as characteristic classes of genus-2 surface bundles [Mo]. Motivated by this interpretation, Cohen completely determined the cohomological structure of Γ_2^0 in [Co; BC]. Since $\mathrm{Sp}(4;\mathbf{Z})$ is the fundamental group of the moduli space of genus-2 abelian varieties, the cohomology $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$ gives rise to characteristic classes of fibrations of these abelian varieties [CL]. Hence, it is an interesting question to determine the structure of $H^*(\mathrm{Sp}(4;\mathbf{Z});\mathbf{Z})$.

Received October 7, 1992. Revision received January 29, 1993.
The second author is partially supported by NSF grant DMS-89-03302.
Michigan Math. J. 41 (1994).