Coarsely Quasi-Homogeneous Circle Packings in the Hyperbolic Plane

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1. Introduction

Let $\mathfrak{J}$ be a triangulation of an open topological disk. By [HS1, Cor. 1.5] (and [Sc]), there is a circle packing $P_{\mathfrak{J}^{(1)}}$ in the complex plane $\mathbf{C}$, unique up to Möbius transformations, whose graph is combinatorially equivalent to the 1-skeleton $\mathfrak{J}^{(1)}$ of $\mathfrak{J}$ and whose carrier is either the unit disk U or the whole plane $\mathbf{C}$. We call the graph $\mathfrak{J}^{(1)}$ *hyperbolic* if the carrier of $P_{\mathfrak{J}^{(1)}}$ is U, and *parabolic* otherwise. For any vertex v in $\mathfrak{J}^{(1)}$, its *valence* is defined to be the number of edges of $\mathfrak{J}^{(1)}$ with an endpoint at v. The graph $\mathfrak{J}^{(1)}$ is said to have *bounded valence* if there is a uniform bound on the valences of its vertices.

The 2-manifold $|\mathfrak{J}|$ is naturally endowed with the unique (singular Riemannian) metric so that every 2-simplex is isometric to a unit equilateral triangle in the Euclidean plane. With this metric, there is also a well-defined conformal structure in the manifold. By Koebe's uniformization theorem, $|\mathfrak{J}|$ is conformally equivalent to either U or $\mathbf{C}$. When $\mathfrak{J}^{(1)}$ has bounded valence, the ring lemma of [RS] (see Lemma 2.2 below) implies that $|\mathfrak{J}|$ is conformally equivalent to U if and only if $\mathfrak{J}^{(1)}$ is hyperbolic. In the following, we will consider $|\mathfrak{J}|$ as both a metric space and as a Riemann surface.

Let $K \geq 1$ be a constant. A (not necessarily continuous) map $f: X \to Y$ between two metric spaces is called a *coarse K-quasi-isometry* if for each pair of points u and v in X,

$$\frac{d(u,v)}{K} - K \leq d(h(u), h(v)) \leq Kd(u,v) + K, \tag{1.1}$$

and if for each point w in Y there is some u in X such that $d(w, f(u)) \leq K$, where (by an abuse of notation) we have used d to denote the metrics in both X and Y. A metric space X will be called *coarsely quasi-homogeneous* if there is some $K \geq 1$ such that for each pair of points u and v in X there is a coarse K-quasi-isometry $h: X \to X$ with $h(u) = v$. Our main theorem is as follows.

THEOREM 1.1. *Let $\mathfrak{J}$ be a triangulation of an open topological disk such that $\mathfrak{J}^{(1)}$ is hyperbolic and of bounded valence, and let $P_{\mathfrak{J}^{(1)}}$ be a circle packing whose carrier is U and whose graph is combinatorially equivalent to $\mathfrak{J}^{(1)}$.*

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