Divergence-Free Vector Wavelets

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1. Introduction

Wavelet decompositions have found a number of applications in recent years, particularly in harmonic analysis, quantum field theory, and signal analysis. In close parallel with this new development has been the construction of bases of wavelets satisfying regularity and decay conditions of one kind or another [1; 5; 10; 12]. Now there seems to be a growing interest in wavelet decompositions tailored to specific problems – a basis of wavelets for a space of functions defined by some differential constraint, for example. Our own interest is in constructing such orthonormal bases in the space of vector fields with vanishing divergence. Such bases should be useful in several contexts, including the study of incompressible fluids [7].

Since the divergence-free condition is invariant with respect to scaling and translation, one might suppose the construction of such wavelets to be straightforward. Surprisingly enough, this does not seem to be the case, except in two dimensions [3]. The 3-dimensional case is already a complicated case – far from trivial – and we solve the problem here. We construct a divergence-free wavelet orthonormal basis, where the wavelets are class C^N with exponential decay – ideal for many applications.

We are currently purusing the 4-dimensional case because we expect space-time wavelets satisfying the continuity equation to be useful in the analysis of conserved currents. As far as other directions are concerned, one can consider other regularity and decay properties. For example, our method of construction may extend to divergence-free Meyer wavelets (i.e., wavelets with Fourier transform in C_0^∞), but we have not pursued that possibility. Nor have we considered the construction of C_0^N wavelets with vanishing divergence. Lemarié has constructed non-orthonormal divergence-free wavelet bases of compact support [11].

Let I be a finite index set, and for each t in I let ψ_t be a vector-valued function defined on 3-dimensional real space. For $\alpha=(r,n,t)$ in $Z\times Z^3\times I$, the wavelet ψ_α is defined by

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