Lusin's Condition (N) and Mappings with Nonnegative Jacobians

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1. Introduction

Let G be a domain in $\mathbf{R}^n$. A continuous mapping $f: G \to \mathbf{R}^n$ is said to satisfy Lusin's condition (N) if $|f(A)| = 0$ whenever $A \subset G$ and $|A| = 0$. Here, $|A|$ denotes the Lebesgue measure of A. Smooth mappings, say C^1, locally Lipschitzian or even continuous mappings in the Sobolev space $W^{1,p}(G, \mathbf{R}^n)$, $p > n$, satisfy condition (N). However, it is well known that for $p < n$, such mappings fail to satisfy condition (N) (cf. [Po]). For $n = 1$ the condition (N) is well understood (see [Sa]), but for $n \geq 2$ necessary and sufficient analytic conditions for (N) are not so clear. The most important recent studies in this area have been made by Reshetnyak [R3] who proved that a quasiregular mapping $f: G \to \mathbf{R}^n$ satisfies the condition (N). For continuous mappings in $W^{1,n}(G, \mathbf{R}^n)$ he also gave, in [R2], a topological condition that implies condition (N).

Because of the importance of condition (N) in applications, in this paper we investigate how this property is related to mappings in the Sobolev space $W^{1,n}(G, \mathbf{R}^n)$. We focus our attention on this class of mappings because of its application to quasiregular mappings and nonlinear elasticity (cf. [Ba] and [Mu]). We use multiplicity functions, defined in terms of topological degree and related to the topological condition given by Reshetnyak; to characterize those mappings that satisfy condition (N) (see Theorem 3.10). We also introduce Sard's condition (SA): A continuous mapping $f: G \to \mathbf{R}^n$ with partial derivatives a.e. satisfies the condition (SA) if $Jf(x) = 0$ a.e. in an open set $A \subset G$ yields $|f(A)| = 0$. This condition can be regarded as a weak counterpart of the Sard-type result for mappings f with rank $f'(x) < n$: If $f: G \to \mathbf{R}^n$ is C^1 then $|f(A)| = 0$ for $A = \{x \in G : Jf(x) = 0\}$. In general, continuous mappings in $W^{1,n}(G, \mathbf{R}^n)$ that satisfy condition (SA) do not satisfy (N); however, if a continuous map $f \in W^{1,n}(G, \mathbf{R}^n)$ has the property that Jf is of one sign almost everywhere, then we show that conditions (N) and (SA) are equivalent (Theorem 3.12). It is easy to see that a quasiregular mapping satisfies (SA), and hence the condition (N) for quasiregular mappings fol lows from the above result.

Received March 18, 1991. Revision received July 8, 1991.
*Research supported in part by a grant from the National Science Foundation.
Michigan Math. J. 39 (1992).