A Mean-Value Theorem for Character Sums

J. FRIEDLANDER & H. IWANIEC

The classical mean-value theorem for Dirichlet polynomials asserts that

$$S(\lambda)=\frac{1}{\phi(q)}\sum_{\chi(q)}\Bigl|\sum_{n\le N}\lambda_n\chi(n)\Bigr|^2=(1+O(q^{-1}N))\|\lambda\|, \tag{1}$$

where

$$\|\lambda\|=\sum_{\substack{n\le N\\(n,q)=1}}|\lambda_n|^2$$

and the implied constant is absolute; cf. [M, Thm. 6.2], where also the result is refined. This result is best possible when $N<q$, in which case the result is true without the error term. In this paper we consider the case $N>q$, and improve on the result for special sequences. We consider convolutions $\lambda=\alpha*\beta*\gamma$ and $N=KLM$, where $\alpha=(\alpha_k)_{k\le K}$, $\beta=(\beta_l)_{l\le L}$, and $\gamma=(\gamma_m)_{m\le M}$, with α_k,β_l arbitrary and $\gamma_m=1$.

Let

$$S^*(\lambda)=\frac{1}{\phi(q)}\sum_{\substack{\chi(q)\\ \chi\ne\chi_0}}\Bigl|\sum_{n\le N}\lambda_n\chi(n)\Bigr|^2.$$

THEOREM 1. *We have*

$$S^*\ll\|\alpha\|\|\beta\|\|\gamma\|[1+q^{-3/4}(K+L)^{1/4}(KL)^{5/4}+q^{-1}(KL)^{7/4}]q^{\epsilon}.$$

REMARKS. Actually, by using (1) and restricting $M\ll q^{1/2}$ (without loss of generality), the term $q^{-1}(KL)^{7/4}$ can be dropped. In this paper, in order to keep the exposition clear, we present only the proof of a special case which does, however, contain all of the basic ideas.

This work was originally motivated by applications to character sums and Dirichlet L-functions. The Pólya–Vinogradov theorem [P; V] gave the first significant estimates in this area:

$$\sum_{m\le M}\chi(m)\ll q^{1/2}\log q$$

so that

Received January 28, 1991.
Michigan Math. J. 39 (1992).