Boundary Behavior of Derivatives of Analytic Functions

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Introduction

In this paper we study the boundary behavior of functions of the form $(1-|z|^2)^n f^{(n)}(z)$; here f is an analytic function defined on the open unit disk D in the complex plane and n is a positive integer. Many well-studied classes of analytic functions arise from requiring functions of the above form to have a certain growth rate. For example, the Bloch space $\mathcal{B}$ is defined to be the set of analytic functions f on D such that $(1-|z|^2)f'(z)$ is bounded on D, and the little Bloch space $\mathcal{B}_0$ is defined to be the set of analytic functions f on D such that $(1-|z|^2)f'(z) \to 0$ as $|z| \to 1$ (of course, z is restricted to the values in D).

Let dA denote the usual Lebesgue area measure on the complex plane. For $p \in [1, \infty)$, the Bergman space L^p_a is defined to be the set of analytic functions f on D such that

$$\int_D |f|^p \, dA < \infty.$$

As is well known (e.g., see [2, Prop. 1.7]), if $f \in L^1_a$ then

$$f(z) = \frac{1}{\pi} \int_D \frac{f(w)}{(1-z\bar{w})^2} \, dA(w) \quad \text{for every } z \in D. \tag{1}$$

Equation (1) suggests that for $f \in L^1(D, dA)$ (not necessarily analytic), we define an analytic function $P(f)$ on D by

$$P(f)(z) = \frac{1}{\pi} \int_D \frac{f(w)}{(1-z\bar{w})^2} \, dA(w).$$

It is useful to know the image under P of certain natural spaces. We begin by noting that P restricted to $L^2(D, dA)$ is the orthogonal projection of $L^2(D, dA)$ onto L^2_a; furthermore, if $p \in (1, \infty)$ then P restricted to $L^p(D, dA)$ is a bounded projection of $L^p(D, dA)$ onto L^p_a (see Theorem 1.10 of [2]; throughout this paper, we choose references most suited to our approach, so the references are not necessarily to the original source).

Received August 20, 1990.
Both authors were partially supported by the National Science Foundation.
Michigan Math. J. 39 (1992).