A New Proof of the Bott–Samelson Theorem

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Let (M, g) be a compact connected Riemannian manifold with $\dim M = n \geq 2$. Let i_M and d_M denote the injectivity radius and the diameter of M. Given any point p in M, let i_p and $C(p)$ denote the injectivity radius at p and the cut locus of p. We shall call M an S_l-*manifold* if, for some point $p \in M$, $C(p)$ is an l-dimensional submanifold of M. If, for every point $p \in M$, $C(p)$ is an l-dimensional submanifold, then we shall call M an ES_l-*manifold*. For example, according to the Allamigeon–Warner theorem [B], all Blaschke manifolds are ES_l-manifolds. In particular, all compact symmetric spaces of rank one (CROSSes) are ES_l-manifolds for some l. The Bott–Samelson theorem [B] can be stated in the following way.

THEOREM. *The integral cohomology ring of a Blaschke manifold is the same as that of a CROSS.*

The purpose of this note is to give a new proof of this theorem by using the Thom isomorphism theorem. More precisely, we shall prove the following two theorems.

THEOREM A. *If M is an ES_l-manifold with $l = 0$, then M is isometric to the standard unit sphere S^n up to a constant factor.*

THEOREM B. *The integral cohomology ring of an S_l-manifold M is the same as that of a CROSS. More precisely, $\pi_1(M) = 0$ or $\mathbf{Z}_2$.*

(1) *$\pi_1(M) = \mathbf{Z}_2$ if and only if $l = n-1$. In this case, M has the homotopy type of $\mathbf{R}P^n$.*
(2) *If $\pi_1(M)$ is trivial, one has only the following possibilities:*
  (a) *$l = 0$, and M is homeomorphic to S^n;*
  (b) *$n = 2m$, $l = n-2$, and M has the homotopy type of $\mathbf{C}P^m$;*
  (c) *$n = 4m$, $l = n-4$, and M has the integral cohomology ring of $\mathbf{H}P^m$;*
  (d) *$n = 16$, $l = 8$, and M has the integral cohomology ring of $\mathbf{Ca}P^2$.*

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