Symmetric Properties of Eigenfunctions of the Laplace Operator on Compact Riemannian Manifolds

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Introduction

In this paper, we deal with amplitude symmetric properties of eigenfunctions of the Laplace operator on compact Riemannian manifolds. We prove that all the eigenfunctions of the Laplace operator on compact Riemannian manifolds with nonnegative Ricci curvature have a certain amplitude symmetry, and that appropriate conditions on the Ricci curvature and the volume of the manifolds yield strong amplitude symmetries of the first eigenfunctions.

Let M be a compact Riemannian manifold, Δ the Laplace operator acting on smooth functions on M, and $\lambda_1 > 0$ the first eigenvalue of Δ. It is well known that

$$\sup_M u > 0 \quad \text{and} \quad \inf_M u < 0$$

for every eigenfunction u corresponding to λ_1. When the equality

$$\sup_M u = -\inf_M u$$

holds for an eigenfunction u corresponding to λ_1, it is usually easier to estimate λ_1 in terms of geometric quantities of M (see [4], [6], and [9]). In [9], Yang and Zhong asked if the above equality holds for all eigenfunctions u corresponding to λ_1 when M has nonnegative Ricci curvature. The following example gives a negative answer to this question.

Consider the real projective space $\mathbf{RP}^2$ with the standard unit sphere

$$\mathbf{S}^2 = \{(x,y,z) \in \mathbf{R}^3; x^2+y^2+z^2 = 1\}$$

as its Riemannian covering space. Clearly, $\mathbf{RP}^2$ has positive Ricci curvature. The function f on $\mathbf{S}^2$ defined by

$$f(x,y,z) = z^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 = \tfrac{3}{2}z^2 - \tfrac{1}{2}$$

induces a function u on $\mathbf{RP}^2$, and u is an eigenfunction corresponding to λ_1 of $\mathbf{RP}^2$ (see, e.g., [2] for the proof of this claim). Direct calculation gives

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