Some Results on BMOH and VMOH on Riemann Surfaces

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1. Introduction

The uniformization theorem for hyperbolic Riemann surfaces states that, given a Riemann surface R, there exist both a Fuchsian group Γ acting on the unit disk Δ and an analytic function $\phi: \Delta \to R$ such that ϕ is an automorphic function relative to the group Γ; that is, $\phi(T(z)) = \phi(z)$ for each $z \in \Delta$ and each $T \in \Gamma$ (see, e.g., [4, Theorem, p. 209]). If we start with a Riemann surface R which possesses a Green's function and a function f analytic on R, we say that $f \in \text{BMOA}(R)$ if

$$\sup_{\lambda \in R} \iint_R |f'(w)|^2 G_R(w, \lambda)\, dA(w) < \infty,$$

where $G_R(w, \lambda)$ is the Green's function on R with singularity at λ and $dA(w)$ denotes the element of area on R. We may also define the analytic function $f_* = f \circ \phi$ on Δ. Here, f_* is an automorphic function on Δ. We say that $f_* \in \text{BMOA}(\Delta/\Gamma)$ if

$$\sup_{w \in F} \iint_F |f_*'(z)|^2 G_R(\phi(z), \phi(w))\, dA(z) < \infty,$$

where

$$F = \{z \in \Delta : |z| \le |T(z)| \text{ for each } T \in \Gamma\}$$

is the so-called Ford fundamental region for the group Γ and where $dA(z)$ is the element of Euclidean area in Δ. The set F, also known as the Dirichlet polygon, is a fundamental set for the group Γ, together with some additional boundary points for this fundamental set. Although a wide variety of choices for a fundamental region are possible, it will avoid a number of difficulties to deal only with this normalized fundamental region.

For $\lambda \in R$, let a be a point in Δ such that $\phi(a) = \lambda$, and define $G_\Gamma(z, a) = G_R(\phi(z), \lambda)$. By a result of Myrberg,

$$G_\Gamma(z, a) = \sum_{T \in \Gamma} \log \left| \frac{1 - \overline{T(a)}z}{z - T(a)} \right| \quad \text{for } z, a \in F$$

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