Bounded Group Actions on Trees and Hyperbolic and Lyndon Length Functions

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Introduction

In [6] it is shown that if a group G acting as a group of isometries on a tree T has a normal subgroup K with bounded action, then the factor group G/K acts on a quotient tree T/K. This paper considers possibilities for K and relationships between the two actions.

A group action gives an associated hyperbolic length function, defined in [1] and [5], and also for each point of the tree a Lyndon length function, defined originally in [4]. In Section 1 it is shown that the hyperbolic length function on G is determined by that on G/K, and in Section 2 it is shown that the Lyndon length functions on G are determined up to equivalence by those on G/K. In Theorem 2.2 it is established that the normal subgroups K with bounded action are contained in subgroups of G defined by Lyndon length functions, and that there is a maximal K. In Section 3, under the assumption that not every element of G has a fixed point, it is shown that (with K maximal) G/K is isomorphic to a subgroup of the additive reals or has a trivial centre.

1. Bounded Actions and Hyperbolic Lengths

Let a group G act as a group of isometries on a metric tree (or **R**-tree) T, equipped with a metric d. The following notation and properties are recalled from [6], where more detail may be found.

A metric tree T has the property that, for any two points $u, v \in T$, there is a unique isometry $\alpha: [0,r] \to T$ with $\alpha(0)=u$ and $\alpha(r)=v$, where $r=d(u,v)$. The image $\alpha([0,r])$ is denoted by $[u,v]$ and is called a *segment* of T.

For each $u \in T$, a Lyndon length function $\ell_u: G \to \mathbf{R}$ is defined by $\ell_u(x) = d(u,xu)$. The set N of non-Archimedean elements of G consists of elements x such that $\ell_u(x^2) \le \ell_u(x)$ for some (and hence all) $u \in T$. It is shown in [6, Prop. 2.2] that $x \in N$ if and only if it fixes some point of T.

A subgroup K of G has *bounded action* on T if, for some (and hence each) $u \in T$, the set of lengths $\{\ell_u(x); x \in K\}$ is bounded. Theorem 3.2 of [6] states that K has bounded action if and only if it fixes some point of T. If a normal

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