Continuity Properties of Selectors and Michael's Theorem

KRZYSZTOF PRZESŁAWSKI & DAVID YOST

0. Introduction

Recall that if A and B are closed, bounded, convex, nonempty subsets of X (i.e., $A, B \in H(X)$), then the Hausdorff distance between A and B is defined by $d_H(A,B) = \sup(\{d(x,A): x \in B\} \cup \{d(x,B): x \in A\})$. Note that X is always isometric to a subset of $H(X)$ if we identify a point x with the singleton set $\{x\}$. Since no ambiguity arises, we will usually write $d(A,B)$ rather than $d_H(A,B)$. We let $K(X)$ denote the subfamily of compact sets in $H(X)$, also equipped with the Hausdorff metric. Of course $K(X) = H(X)$ when X is finite-dimensional.

Michael's selection theorem [37, Thm. 3.2″] tells us that there is a continuous map $f: H(X) \to X$ such that $f(A) \in A$ for all $A \in H(X)$. For a given Banach space X, can we find a Lipschitz map $f: H(X) \to X$ satisfying the same selection identity? We will refer to selection maps from $H(X)$ to X as *selectors*.

Various authors [40; 43; 45] have observed that this is possible if X is finite-dimensional. Indeed, the Steiner point [46] provides a suitable selector when $X = \mathbf{R}^n$. It is noted in [55] that this is not possible if $X = C[0,1]$. The arguments used in these papers are reasonably elementary.

In this paper, we concern ourselves with the existence of uniformly continuous selectors for general Banach spaces. It follows from [32, Cor. 5] that there is no uniformly continuous selector from $H(X)$ to X whenever X is an infinite-dimensional Hilbert space. This depends on various results which may be found in [24; 32; 38]. From Dvoretzky's theorem it can then be deduced that there is no uniformly continuous selector from $H(X)$ (or even from $K(X)$) to X, whenever X is an infinite-dimensional Banach space. This result could have been proved twenty years ago. Although a special case has already been published [16, p. 245], it does not seem to be very well known. We feel that this problem deserves a thorough exposition. Two new proofs will be presented here, in Sections 2 and 3, indicating how this problem interacts with different areas of analysis.

The original proof of [32, Cor. 5] depended upon a result of Lindenstrauss [32] that if a closed subspace M is the range of a uniformly continuous re-

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