AN EXTENSION OF WIDDER'S THEOREM

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1. Introduction. In this paper we consider a problem concerning the boundary behavior of solutions of the one-dimensional heat equation on the strip (or the half-plane) $\mathfrak{D}_c = \mathbf{R} \times (0, c)$, where $0 < c \leq +\infty$. By a solution of the heat equation on an open set $\mathfrak{D} \subseteq \mathbf{R}^2$ we understand here a twice continuously differentiable real function $u(x,t)$, $(x,t) \in \mathfrak{D}$, such that $u_{xx} = u_t$ in $\mathfrak{D}$.

It is well known that many properties of such functions are similar to those of harmonic functions (see e.g. [8], [6], [3], [4], and [2]). One of these similarities is that nonnegative harmonic function on $\mathfrak{D}_\infty$ and nonnegative solutions of the heat equation on $\mathfrak{D}_c$ both have Poisson-type integral representations. In the "harmonic" case this fact is attributed to F. Riesz and Herglotz, and in the case of solutions of the heat equation it is a theorem due to Widder [8]. In [5] Hayman and Korenblum obtained "an extension of the Riesz–Herlotz formula" by showing that for a continuous positive nonincreasing function $k(t)$, $t > 0$, the condition

$$\int_0^1 \sqrt{k(t)/t}\, dt < +\infty$$

is equivalent to the property that each harmonic function h defined on $\mathfrak{D}_\infty$, with $h(x,t) \leq k(t)$, $t > 0$, can be represented in the form

$$h(x,t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{t}{(x-y)^2 + t^2}\, d\left(\lim_{\tau \to 0+} \int_0^y h(z,\tau)\, dz \right) + Ct.$$

The outer integral in the above formula was originally defined by the integration-by-parts formula, but, as shown later in [7], it can be understood as a Riemann–Stieltjes integral (with respect to a function which may not necessarily be of bounded variation). The aim of this paper is to show an analogue of that result for solutions of the heat equation on $\mathfrak{D}_c$.

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2. Main results. Let $K(x,t)$ be the Gauss kernel, that is,

$$K(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t} \right), \quad x \in R, \ t > 0.$$

In the sequel k will always denote a positive nonincreasing unbounded continuous function on $(0, +\infty)$.

THEOREM 1. *Let* $\lim_{t \to 0+} \sqrt{t}\, k(t) = 0$ *and*

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