SOME KREIN SPACES OF ANALYTIC FUNCTIONS AND AN INVERSE SCATTERING PROBLEM

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1. Introduction. The present paper deals with certain reproducing kernel Krein spaces (RKKS) of analytic functions, and we first recall the definition of a reproducing kernel Hilbert space (RKHS). A Hilbert space $\mathcal{H}$ of $\mathbf{C}_n$-valued functions defined on some set S has reproducing kernel $K_\omega(\lambda)$, where $K_\omega(\lambda)$ is a $\mathbf{C}_{n\times n}$-valued function defined for λ and ω in S if

(1) for any ω in S and c in $\mathbf{C}_n$, the function $K_\omega c\colon \lambda \mapsto K_\omega(\lambda)c$ belongs to $\mathcal{H}$;

(2) for any f in $\mathcal{H}$, ω in S, and c in $\mathbf{C}_n$,

$$\langle f, K_\omega c\rangle = c^* f(\omega), \tag{1.1}$$

where $\langle\,,\rangle$ denotes the inner product in $\mathcal{H}$.

The function $K_\omega(\lambda)$ is easily seen to be unique and symmetric:

$$K_\omega(\lambda) = K_\lambda^*(\omega). \tag{1.2}$$

It is moreover positive: for any r, any choice of r points in S, $\omega_1, \dots, \omega_r$, and any $c_1, \dots, c_r$ in $\mathbf{C}_n$, the $r\times r$ matrix with ij entry

$$c_j^* K_{\omega_i}(\omega_j) c_i \tag{1.3}$$

is positive.

Conversely, by the matrix version of a result of Moore [5], to any $\mathbf{C}_{n\times n}$-valued function $K_\omega(\lambda)$ defined on some set S and positive in the sense just explained one can associate a unique reproducing kernel Hilbert space of $\mathbf{C}_n$-valued functions defined on S with reproducing kernel $K_\omega(\lambda)$.

In [19], Sorjonen relaxed the positivity condition and supposed that the function $K_\omega(\lambda)$ has v negative squares. Under this weaker hypothesis, the result of Moore may be extended. There exists a unique reproducing kernel Pontryagin space of $\mathbf{C}_n$-valued functions defined on S with reproducing kernel $K_\omega(\lambda)$.

The problem of associating to a given $\mathbf{C}_{n\times n}$-valued function $K_\omega(\lambda)$ subject to (1.2) a reproducing kernel Krein space with reproducing kernel $K_\omega(\lambda)$ seems open, and the aim of the present paper is twofold: first, in Section 2, we construct a reproducing kernel Krein space when $K_\omega(\lambda)$ is of the form

$$\frac{X(\lambda)JX^*(\omega)}{\rho_\omega(\lambda)}, \tag{1.4}$$

where X is a $\mathbf{C}_{n\times m}$-valued function of bounded type in Δ_+ (Δ_+ designates either the open unit disk $\mathbf{D}$ or the open upper half plane $\mathbf{C}_+$), where J is a signature matrix, that is, a matrix subject to

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