CLUSTER SET, ESSENTIAL RANGE, AND DISTANCE ESTIMATES IN BMO

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To my teacher, Allen Shields, for his sixtieth birthday

Introduction. This paper focuses primarily on functions F, holomorphic or harmonic on the open unit disc U, for which the radial limit function

$$F^*(\zeta) = \lim_{r \to 1-} F(r\zeta)$$

exists finitely for almost every $\zeta \in \partial U$. For example, F could be a holomorphic function of bounded characteristic, or it could be the Poisson integral of a measure on the boundary. Our goal is to study the relationship between the *cluster set* of F and the *essential range* of F^*.

Clearly the cluster set contains the essential range. We want to know when they coincide. Obviously they do if F extends continuously to the boundary of U, but consideration of the "unit singular function" $F(z) = \exp\{(z+1)/(z-1)\}$ shows that they need not, even if F is bounded and holomorphic on U.

We are going to prove that cluster set and essential range coincide whenever F is the Poisson integral of a function of *vanishing mean oscillation*. This class contains all harmonic functions which extend continuously to the boundary, some which do not, and even some which are unbounded. Our result shows that every function of vanishing mean oscillation has connected essential range; it recovers the well-known fact that among the inner functions, only the finite Blaschke products can have boundary function of vanishing mean oscillation [21, §3]; and it has consequences for the algebra QC of quasi-continuous functions on the unit circle.

These results emerge from a distance estimate: *If F is the Poisson integral of a function of bounded mean oscillation, then the distance from F^* to the space of functions of vanishing mean oscillation is bounded below by the Hausdorff distance between the cluster set and the essential range* (in plain English: the largest distance by which you can avoid every point of the essential range, while staying in the cluster set). Examples show that this estimate is sharp.

Our proofs work as well for the unit ball B of $\mathbf{C}^n$ with $n > 1$. A feature peculiar to higher dimensions is the fact that the sets on which a holomorphic function has constant value "propagate" to the boundary: every value is a cluster value. Our results therefore show that if F, holomorphic on B, is the Poisson integral of a function of vanishing mean oscillation on ∂B, then the essential range of F^* contains $F(B)$.

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