PRIME IDEALS IN CLOSED SUBALGEBRAS OF L^∞

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Let $\mathbf{D}$ denote the open disc and let H^∞ denote the algebra of bounded analytic functions on $\mathbf{D}$. A prime ideal in a commutative algebra A is an ideal Q such that whenever $f, g \in A$ and $fg \in Q$, either $f \in Q$ or $g \in Q$. In [11, p. 396] the following question is asked: Let Q be a nonzero prime ideal in H^∞ such that $Q \neq H^\infty$, and suppose Q is finitely generated; do we then have $Q = \{f \in H^\infty : f(\zeta) = 0\}$, where $\zeta \in \mathbf{D}$? In the first section of this paper, we shall answer this question affirmatively. After this work was completed, I learned that R. Mortini also obtained this result ([14], [15]).

Let C denote the algebra of continuous functions on the unit circle, $\partial\mathbf{D}$. In §2, we shall show that $H^\infty + C$ has no nontrivial finitely generated prime ideals. However, there do exist proper closed subalgebras of L^∞ which have nontrivial finitely generated prime ideals.

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1. Finitely generated prime ideals in H^∞. Let B be a closed subalgebra of L^∞. The maximal ideal space of B is denoted $M(B)$. By maximal ideal we mean a proper ideal of B contained in no other proper ideal of B. Because each such ideal is the kernel of a nonzero complex homomorphism on B, we think of $M(B)$ as the space of nonzero complex homomorphisms on B. With the weak-*topology, $M(B)$ is a compact Hausdorff space. In the usual way, we think of $\mathbf{D}$ as a subset of $M(H^\infty)$. By the Corona theorem, $\mathbf{D}$ is a dense subset of $M(H^\infty)$. If B contains H^∞, then the space $M(B)$ can be identified with a closed subset of $M(H^\infty)$. If B properly contains H^∞, then B contains $H^\infty + C$. Thus $M(B) \subseteq M(H^\infty) - \mathbf{D}$ $(= M(H^\infty + C))$. We shall identify a function in B with its Gelfand transform.

In this section, our main tool is the analytic structure of the Gleason parts of H^∞. For $x \in M(H^\infty)$, the Gleason part containing x is denoted $P(x)$. If f denotes a function in H^∞ and $x \in M(H^\infty)$ is such that $f(x) = 0$, then the order of the zero of f at x is the supremum of the positive integers n such that f can be factored as $f = f_1 \cdots f_n$, $f_j \in H^\infty$ and $f_j(x) = 0$ for $j = 1, 2, \ldots, n$. The order of the zero of f at x will be denoted by Ord $Z(f; x)$. The zero set f in $M(B)$ is denoted $Z_B(f)$. We shall also use the following lemma (usually referred to as Nakayama's lemma).

LEMMA 1.1 [12, p. 11]. *Let A be a commutative ring with identity, M a finitely generated A module and J an ideal of A. Suppose that $JM = M$. Then there exists an element $a \in A$ of the form $a = 1 + b$, $b \in J$, such that $aM = 0$.*

We shall apply Lemma 1.1 to the case in which $A = H^\infty$ and M is a finitely generated ideal of A. Since H^∞ has no zero divisors, if we produce a proper ideal J such that $JM = M$, our conclusion is that $M = 0$.

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