CHARACTERIZING CERTAIN INCOMPLETE INFINITE-DIMENSIONAL ABSOLUTE RETRACTS

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0. Introduction and preliminaries. The study of infinite-dimensional manifolds modeled on $Q=[-1,1]^{\infty}$ and $s=(-1,1)^{\infty}$ reached a climax when H. Torunczyk gave a topological characterization theorem for these spaces: A locally compact ANR is a Q-manifold if and only if any map $f: C \to X$ of a compact (metric) space can be approximated by a closed embedding. Similarly, a complete ANR X is an s-manifold if and only if any map $f: C \to X$ of a complete (metric) space can be approximated by a closed embedding.

The second author has characterized manifolds modeled on $\sigma=\{(t_1, t_2, \ldots) \in [-1,1]^{\infty}: t_i=0$ for all but finitely many $i\}$ and $\Sigma=\{(t_1, t_2, \ldots) \in Q^{\infty}: t_i=0$ for all but finitely many $i\}$ in the same spirit [20]: An ANR X is a σ-manifold if and only if X can be represented as a countable union of finite-dimensional compacta, each of which is a strong Z-set in X, and any map $f: C \to X$ of a finite-dimensional compactum C, that is a Z-embedding when restricted to a closed subset $D \subseteq C$, can be approximated by a Z-embedding $g: C \to X$ so that $g \mid D = f \mid D$. (The characterization theorem for Σ-manifolds is obtained by deleting the words "finite-dimensional.") Although the resemblance with the characterization theorems for Q-manifolds and s-manifolds is obvious, one cannot avoid observing the much cleaner structure of Torunczyk's theorems. However, the mention of strong Z-sets is necessary, since examples of fake s-manifolds constructed in [4] lead to a straightforward construction of an AR X that can be represented as $\sigma \cup \{\text{point}\}$, such that $X \not\approx \sigma$, but X satisfies the hypotheses of the characterization theorem for σ, after deleting the word "strong." Similarly, if we replace the relative approximation condition by an absolute one (i.e., requiring $D=\varnothing$), then a counterexample is constructed by J. P. Henderson and J. J. Walsh [18].

In this paper we introduce a notion of strong $\mathcal{C}$-universality for a class $\mathcal{C}$ of (separable, metric) spaces. In the case that $\mathcal{C}=\{(\text{finite-dimensional})\text{ compacta}\}$ this is precisely the property stated in the characterization theorem for Σ (respectively σ).

The key idea that allows one to prove the characterization theorem for Σ and σ is the notion of an (f.d.) cap set (finite-dimensional compact absorption set), due to R. D. Anderson [2]. Loosely speaking, $\Sigma \cong Q-s \subset Q$ is a cap set, since it is strongly $\mathcal{C}$-universal ($\mathcal{C}=\{\text{compacta}\}$) and there are small maps $Q \to \Sigma \subset Q$. This notion has been subsequently generalized by different authors (cf. [5], [24], [27], [14]). In §3 we introduce the definition of a $\mathcal{C}$-absorbing set, which represents

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