ON THE DIAGONALIZATION OF A CERTAIN CLASS OF OPERATORS

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1. Introduction and results. We consider infinite dimensional matrices of bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$. The space $\mathcal{H}$ is assumed to have a suitable orthonormal basis, with respect to which the matrix elements are defined. We shall also perform formal operations on the matrix elements to form new matrices, which do not necessarily correspond to bounded operators without extra assumptions. Unless it is clear enough from the context, we will indicate if a matrix is related to an operator or if it should be understood only as a formal array of complex numbers.

First we give some definitions. Let $H=(h_{ij})$, $i,j=1,2,\ldots$, be an infinite-dimensional matrix of an operator. We are concerned about finding, under suitable assumptions, an invertible matrix U such that UHU^{-1} is diagonal. We look for U in the form $U=\exp A$. If H is hermitian, then A will be antihermitian and U unitary.

We write H^* for the adjoint of H, that is, $H^*_{ij}=\bar{h}_{ji}$. We define the matrix H^D by

$$H^D_{ij}=\begin{cases} h_{ij}, & i=j, \\ 0, & i\neq j, \end{cases}$$

and set $H^K=H-H^D$. If B is a matrix and $b_{ii}\neq b_{jj}$ for $i\neq j$, we define H/B as the formal matrix whose entries are given by

$$(H/B)_{ij}\equiv\left(\frac{H}{\bar{B}}\right)_{ij}=\frac{h_{ij}}{b_{ii}-b_{jj}}, \quad i\neq j,$$
$$=0, \qquad i=j.$$

We note that H/B does not depend on B^K or on H^D.

The commutator of the matrices A and B is $[A,B]=AB-BA$, provided that the products AB and BA are well defined. We have

$$[H/B, B^D]=-H^K. \tag{1.1}$$

The Hilbert–Schmidt norm $\|H\|_2$ of H is given by

$$\|H\|_2^2=\sum_{i,j=1}^{\infty}|h_{ij}|^2,$$

and the operator norm $\|H\|$ by

$$\|H\|=\sup\{\|Hx\| \mid \|x\|=1\}.$$

Here $\|x\|^2=\sum_{i=1}^{\infty}|x_i|^2$ for $x\in\mathcal{H}$.

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