ON THE SINGULARITIES OF SIMPLE PLANE CURVES

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Let Γ be a differentiable curve in a real projective plane P^2 met by any line in P^2 at a finite number of points. The singular points of Γ are inflections, cusps (cusps of the first kind), and beaks (cusps of the second kind). Let n_1, n_2, and n_3 be the number of these points in Γ respectively. Γ is non-singular if $n(\Gamma)= n_1+n_2+n_3=0$. We are interested in the following questions: Under what conditions is Γ non-singular? What is then the minimum value of $n(\Gamma)$ and how is this minimum value related to n_1, n_2, and n_3? We assume that every line in P^2 meets Γ and thus exclude all non-singular conics.

It is known that either every line in P^2 meets Γ with an odd multiplicity or every line in P^2 meets Γ with an even multiplicity. The curves of the former type have been studied by Möbius [3], Kneser [2], and Scherk [6], among others. In [1], we began an investigation of curves of the latter type. Though there exist such non-singular curves with as few as one multiple point, it was shown that the existence of singularities is dependent not only on the number, but also the type, of multiple points in the curve. More precisely, if a curve Γ is *almost-simple* (any closed subarc of Γ with coincident end-points is met by every line in P^2) then $n(\Gamma)\geq 2$ and $n_1+2n_2+n_3\geq 4$. This investigation is continued in the present paper under the assumption that Γ possesses no multiple points. We claim that $n(\Gamma)\geq 3$ and if $n_2>0$, then $n_1+2n_2+n_3\geq 6$.

We assume that P^2 has the usual topology. Let $p, q, \ldots$ and $L, M, \ldots$ denote the points and lines of P^2 respectively. We denote by $\langle p, L, \ldots\rangle$, the flat of P^2 spanned by $p, L, \ldots$.

Differentiable curves. Let $T\subset P^2$ be an oriented line. For $t_0\neq t_1$ in T, let $[t_0, t_1]$ denote the oriented closed segment of T with the initial point t_0 and the terminal point t_1. We set $(t_0, t_1)=[t_0, t_1]\setminus\{t_0, t_1\}$, $[t_0, t_1)=(t_0, t_1)\cup\{t_0\}$, and $(t_0, t_1]=(t_0, t_1)\cup\{t_1\}$. Then $T=[t_0, t_1]\cup(t_1, t_0)=[t_0, t_1)\cup[t_1, t_0)$. By a (two-sided) neighborhood of t in T we mean a segment $U(t)=(t_0, t_1)$ containing t. Then $U^-(t)=(t_0, t)$, $U^+(t)=(t, t_1)$, and $U'(t)=(t_0, t)\cup(t, t_1)$ are left, right, and deleted neighborhoods of t in T respectively.

A *curve* Γ in P^2 is a continuous map from T into P^2. A line M is the *tangent* of Γ at t if $M=\lim\langle\Gamma(t), \Gamma(t')\rangle$ as t' tends to t in $T\setminus\{t\}$; in which case we set $M=\Gamma_1(t)$. A curve Γ is (*directly*) *differentiable* if $\Gamma_1(t)$ exists for each $t\in T$ and any line in P^2 meets $\Gamma(T)$ at a finite number of points. Henceforth Γ is differentiable, and for convenience we identify $\Gamma(T)$ with Γ.

Let $\mathfrak{M}\subseteq T$ be a segment. We call $\Gamma|_{\mathfrak{M}}$ a *subarc* of Γ and again identify $\Gamma(\mathfrak{M})$ with $\Gamma|_{\mathfrak{M}}$. As $|L\cap\Gamma(\mathfrak{M})|<\infty$ for any L, we say that $\mathfrak{M}$ has *finite order*. If in addition $n=\sup_{L\subset P^2}|L\cap\Gamma(\mathfrak{M})|$ is finite, we say that $\mathfrak{M}$ is of *order* n. The *order*

Received March 17, 1983. Revision received September 25, 1984.
Michigan Math. J. 32 (1985).