TOPOLOGICAL RESULTS IN COMBINATORICS

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Let Δ be a finite simplicial complex with vertex set $V=\{x_0,\ldots,x_n\}$. Let $X=|\Delta|$ denote its underlying topological space. Let K be a field. Associated with Δ and K is a certain ring $K[\Delta]$, described as follows: Let $S=K[x_0,\ldots,x_n]$ be the polynomial ring over K with indeterminates $x_0,\ldots,x_n$. Let I_Δ be the ideal of S generated by all monomials $x_{i_0}\cdots x_{i_r}$ such that $i_0<\cdots<i_r$ and the vertices $x_{i_0},\ldots,x_{i_r}$ do *not* span a simplex of Δ. Define $K[\Delta]=S/I_\Delta$.

Now when $K[\Delta]$ is considered as a module over the polynomial ring S, it has a finite free resolution

$$0\longrightarrow M_j\longrightarrow\cdots\longrightarrow M_1\longrightarrow M_0\longrightarrow K[\Delta]\longrightarrow 0;$$

this is an exact sequence of S-modules, where each M_i is free. Furthermore, there is a unique such graded resolution which minimizes the rank of each M_i; such a resolution is called *minimal*. Let $b_i=b_i(K[\Delta])$ be the rank of the module M_i in this minimal free resolution. The largest integer i for which $b_i\neq 0$ is called the *homological dimension* (or the *depth*) of $K[\Delta]$, and denoted $h(\Delta)$.

It is known that $n-\dim\Delta\leqslant h(\Delta)\leqslant n+1$, where we recall that $n+1$ is the number of vertices of Δ. If $h(\Delta)=n-\dim\Delta$, then the ring $K[\Delta]$, and by extension the complex Δ, is said to be *Cohen-Macaulay.* If this condition is satisfied and if in addition $b_{h(\Delta)}=1$, then the ring and the complex are said to be *Gorenstein.* These conditions have been extensively studied by M. Hochster [1] and R. Stanley [4].

Hochster conjectured that the Cohen-Macaulay condition is independent of the simplicial structure of Δ, depending only on the underlying topological space. This conjecture has turned out to be correct, and in fact was almost proved by a student of Hochster's, G. Reisner. In his thesis [3], Reisner derived a condition involving the links of simplices in Δ, which he proved equivalent to the condition that Δ be Cohen-Macaulay. It requires only a short additional argument to show his condition equivalent to one which is topologically invariant. See Corollary 3.4 following.

A more general conjecture was suggested to the author by Stanley. The Cohen–Macaulay condition is just the condition that the number $n-h(\Delta)-\dim\Delta$ should vanish. Stanley conjectured that *this number itself* is a topological invariant of $|\Delta|$. Our purpose in this paper is to prove this conjecture. It suffices to prove $n-h(\Delta)$ a topological invariant, since it is well-known that $\dim\Delta$ is.

The proof relies on a theorem of Hochster's, stated in §1, which expresses the numbers b_i in terms of the cohomology (with coefficients in K) of Δ and its subcomplexes. In §2 we use Hochster's theorem to give a proof of our conjecture for complexes whose underlying spaces are the sphere S^N and the ball B^N. This case

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