SOME INEQUALITIES FOR THE MODULI OF CURVE FAMILIES

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1. Introduction. The modulus of a curve family is a basic tool in the theory of quasiconformal and quasiregular mappings in $\mathbf{R}^n$. The numerical value of the modulus is known only for a few curve families. Therefore good estimates are of importance. Several estimates are given in the literature ([1], [2], [4], [7], [9]). However, the known estimates are not adequate in all those cases which are relevant to the theory of quasiconformal and quasiregular mappings.

Let $E, F \subset \bar{\mathbf{R}}^n$ be non-empty sets and let $\Delta_{EF} = \Delta(E,F)$ be the family of all closed curves which join E to F in $\bar{\mathbf{R}}^n$. In this paper we shall study the problem of finding estimates for the modulus $M(\Delta_{EF})$ in terms of the 'sizes' of E and F, in particular, when E and F are 'small.' We list some well-known estimates.

(a) E and F are connected (cf. Gehring [1], [2], and Väisälä [9, pp. 27–40]). In this case $M(\Delta_{EF})$ has a lower bound, which depends on the dimension n and the spherical diameters $q(E)$ and $q(F)$.

(b) E is connected and $\operatorname{cap} F > 0$ (i.e., F is of positive conformal capacity). In this case $M(\Delta_{EF})$ has a lower bound, which depends on $q(E)$, F, and n (cf. Martio, Rickman, and Väisälä [4, 3.11]).

(c) $\operatorname{cap} E > 0$ and $\operatorname{cap} F > 0$. In this case $M(\Delta_{EF})$ has a lower bound depending only on E, F, and n (cf. Näkki [7]).

The lower bound in (a) is an increasing function of $\min\{q(E), q(F)\}$. It seems that in the cases (b) and (c) the dependence of the lower bound on the 'sizes' of E and F is more complicated and that a further study of this dependence is desirable. The condition of being of positive capacity in (b) and (c) measures the local structure of the set rather than its global size. In order to achieve a quantitative lower bound for $M(\Delta_{EF})$ also in cases (b) and (c) we introduce a set function $c(\cdot)$ with the following properties.

1.1 THEOREM. *There exists a set function* $c(\cdot)$ *in* $\bar{\mathbf{R}}^n$ *with the properties*

(1) $c(E) = c(hE)$ *whenever* $h: \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ *is a spherically isometric Möbius transformation.*

(2) $c(\cdot)$ *is a quasiadditive* (*for definition cf.* 3.20) *outer measure with* $0 \leqslant c(E) \leqslant c(\bar{\mathbf{R}}^n) < \infty$.

(3) $c(E) > 0$ *if and only if* $\operatorname{cap} E > 0$.

(4) *If E is connected, then* $c(E) \geqslant a_n q(E)$, *where* a_n *is a positive number depending only on* n.

(5) $M(\Delta_{EF}) \geqslant \beta \min\{c(E), c(F)\}$ *where* β *is a positive number depending only on* n.

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