ON CURVATURE AND SIMILARITY

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1. Introduction. The purpose of this note is to shed some light on the relationship between the Cowen–Douglas curvatures $\mathcal{K}_T$ and $\mathcal{K}_S$, for two similar operators T, S of class $B_1(\Omega)$, by making use of recent results on the similarity of Toeplitz operators [1].

To be specific, let Ω be a planar region. We say a bounded operator T on a Hilbert space H belongs to $B_1(\Omega)$ if $T-\lambda I$ is onto and has 1-dimensional kernel for $\lambda \in \Omega$, and if

$$\bigvee_{\lambda\in\Omega} \ker(T-\lambda I)$$

is dense in H. For $T\in B_1(\Omega)$, the curvature $\mathcal{K}_T$ is defined, for $\lambda\in\Omega$, by

$$\mathcal{K}_T(\lambda) = -\frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}\log\|k_\lambda\|^2,$$

where $\{k_\lambda\}$ is an analytic determination of the set of null vectors of $T-\lambda I$, $\lambda\in\Omega$.

In [4], Cowen and Douglas introduce B_1 and $\mathcal{K}_T$ and prove, among other things, that $\mathcal{K}_T$ is a complete unitary invariant for $T\in B_1(\Omega)$. But for similar $S, T\in B_1(\Omega)$, the situation is not made so clear. In fact, the best analogue of the result for unitary equivalent S and T is left as a conjecture for the case of similarity. Let $S, T\in B_1(\mathbf{D})$, $\mathbf{D}$ the unit disk, and suppose the closure $\bar{\mathbf{D}}$ of $\mathbf{D}$ is a k-spectral set for S and T, for some k. The *Cowen–Douglas conjecture* ([4], p. 252) states that *if S and T are similar, then*

$$\lim_{\lambda\to\lambda_0\in\mathbf{T}} \mathcal{K}_T(\lambda)/\mathcal{K}_S(\lambda) = 1,$$

where $\mathbf{T}$ is the unit circle. (Actually, Cowen and Douglas also conjecture the converse statement; we shall have no further comment concerning the converse, however.)

In Section 2, using a "piece" of Toeplitz operator from [1], we show that the Cowen–Douglas conjecture is false. In Section 3, we investigate our example further, showing how the failure of the conjecture can be used to obtain a spectral set estimate. In Section 4, we describe a class of Toeplitz operators for which the Cowen–Douglas conjecture holds.

2. The example. Let T_F denote the Toeplitz operator with symbol

$$F(z) = z^2/(z-\beta) \quad \tfrac{1}{2}<\beta<1,$$

so that, for $x\in H^2$,

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