INVARIANT PSEUDODIFFERENTIAL OPERATORS ON TWO STEP NILPOTENT LIE GROUPS

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Introduction. Let G be a connected, simply connected two step nilpotent Lie group with Lie algebra $\mathcal{G}$. We shall develop a calculus of right invariant pseudodifferential operators on G which is based on the representation theory for G.

For $p \in \mathcal{S}^*(\mathcal{G}^*)$ define an operator $\lambda(p): \mathcal{S}(G) \to \mathcal{S}^*(G)$ by $\lambda(p)\phi = (Fp \circ \log) * \phi$, where $\mathcal{S}^*$ denotes the dual of the space of rapidly decreasing functions, $F: \mathcal{S}^*(\mathcal{G}^*) \to \mathcal{S}^*(\mathcal{G})$ is the Fourier transform, $\log: G \to \mathcal{G}$ is the inverse of the exponential map and $*$ is convolution as determined by the product on G. $\lambda(p)$ is the right invariant pseudodifferential operator with symbol p, $\lambda(p)$ being a partial differential operator if and only if p is a polynomial. Howe [5] showed that for arbitrary nilpotent Lie groups the calculus of invariant pseudodifferential operators "fibers" over $\mathcal{Z}^*$, where $\mathcal{Z}$ is the center of $\mathcal{G}$. In the case of the Heisenberg group this is essentially equivalent to saying that the calculus fibers over the orbits of the coadjoint action of G on $\mathcal{G}^*$. To be more explicit, if p and q are both in $\mathcal{S}(\mathcal{G}^*)$ let $p\#q$ be that element of $\mathcal{S}(\mathcal{G}^*)$ such that $\lambda(p\#q) = \lambda(p)\lambda(q)$. Then $p\#q(\xi)$ depends only on the values of p and q on that orbit of the coadjoint action that contains ξ. Furthermore, as was shown in Howe [4], after making the appropriate identification the calculus at the orbit level is the Weyl pseudodifferential operator calculus. In fact, as we show in Section 1, from this it follows easily that the calculus fibers over the orbits of the coadjoint action and the orbit level calculus is the Weyl calculus for any step two group G. If π is an irreducible unitary representation of G, the Kirillov theory then allows us to define an operator $\pi(p)$ for which the Weyl symbol is essentially p restricted to the orbit corresponding to π, and $\pi(p\#q) = \pi(p)\pi(q)$.

In Sections 2 and 3 it is shown that the calculus can be extended to classes of symbols $S^m(\Phi, \mathcal{G}^*)$, where Φ is some weight function as defined in Section 2, and $p \in S^m(\Phi, \mathcal{G}^*)$ if p satisfies estimates of the form

$$|D_1 \cdots D_k p(\xi)| \leqslant C\Phi(\xi)^{m-k}$$

where each D_j is a vector field which is tangent to the orbits of the coadjoint action. No assumptions are made about the differentiability of p except in directions tangent to the orbits. We prove that if $p \in S^m$ and $q \in S^k$, then $p\#q \in S^{m+k}$, and we give an asymptotic expansion for $p\#q$. In Section 4 it is shown that $\lambda(p)$ is a bounded operator on $L^2(G)$ if $p \in S^0(\Phi, \mathcal{G}^*)$. The main theorems of the calculus are proved by appealing to Hörmander [3] for the corresponding results in the Weyl calculus at the orbit level and showing that certain estimates are uniform over the orbits.

L^2 boundedness has been proved by Howe [5], without appealing to a standard operator calculus, for general nilpotent Lie groups in the case where $\Phi(\xi) = (1 + |\xi|)^\delta$ with $\delta > 1/2$ and where the derivatives of p satisfy the appropriate estimates in all

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