CAUCHY TRANSFORMS AND BEURLING-CARLESON-HAYMAN THIN SETS

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1. INTRODUCTION

Let Δ denote the open unit disc of the complex plane, and T the unit circle. The *Cauchy transform* of a Schwartz distribution μ on T is the function C_μ holomorphic in Δ defined by

$$C_\mu(z) = \sum_{n=0}^{\infty} \hat{\mu}(n) z^n = \langle \mu, k_z \rangle \qquad (z \in \Delta),$$

where k_z is the Cauchy kernel $k_z(\tau) = (1 - \bar{\tau}z)^{-1}$, $(\tau \in T)$ and $\hat{\mu}$ is the Fourier transform of μ [9; Chapter I, sec. 7, Problem 5, pp. 43–44]. In this paper we characterize those closed subsets E of T for which C_μ has bounded characteristic in Δ whenever support $\mu \subset E$.

To get some feeling for this problem, observe that if E is a finite set and support $\mu \subset E$, then C_μ is a linear combination of derivatives of Cauchy kernels, hence simple estimates show that C_μ belongs to the Hardy space H^p for all sufficiently small p. In particular, C_μ has bounded characteristic in Δ. On the other hand, if $E = T$ then there are distributions which are "almost measures" for which C_μ is not of bounded characteristic [8; Chapter X, Prop. 2, p. 110].

Our characterization involves the decomposition of $T \setminus E$ into a countable disjoint union of open sub-arcs (I_n) of T. Letting ϵ_n denote the length of I_n, we can state the main result as follows:

THEOREM. *The following statements about E are equivalent:*

(i) *If μ is a distribution with support contained in E, then C_μ has bounded characteristic in Δ.*

(ii) *E has measure zero and* $\sum \epsilon_n \log \epsilon_n > -\infty$.

The sets E of measure zero for which $\sum \epsilon_n \log \epsilon_n > -\infty$ are usually called *Carleson sets,* and they have an interesting history. They were first introduced by A. Beurling [2], who observed that if a function f is continuous on the closed unit disc, holomorphic in the interior, and satisfies a Lipschitz condition on T, then

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