THE CAUCHY PROBLEM FOR CONVOLUTION OPERATORS. UNIQUENESS

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SECTION 1

In this paper we shall discuss the uniqueness of the Cauchy problem for convolution equations in $\mathbf{R}^{n+1}$. The variables in $\mathbf{R}^{n+1}$ will be denoted by $(x,t) = (x_1, \ldots, x_n, t)$. The dual variables in $\mathbf{C}^{n+1}$ will be denoted by

$$z = (\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta).$$

$\operatorname{Im} \xi$ stands for $(\operatorname{Im} \xi_1, \ldots, \operatorname{Im} \xi_n)$, and similar expressions for $\operatorname{Im} z$, $\operatorname{Re} \xi$, etc. The bracket $\langle z, w \rangle$ denotes the usual bilinear product in $\mathbf{C}^n$ or $\mathbf{C}^{n+1}$, according to the context, *e.g.*, $\langle \xi, x \rangle = \xi_1 x_1 + \ldots + \xi_n x_n$. The closed half-space $\{(x,t) \in \mathbf{R}^{n+1} : t \geq 0\}$ will be denoted by $\mathbf{R}^{n+1}_+$.

All the functions or distributions considered will always depend on $n+1$ variables, unless it is explicitly stated otherwise, *e.g.*, if we write a function Φ as $\Phi(x)$ it means that it depends only on the first n variables.

Let us recall that a convolution operator in the space $\mathscr{D}'$ of distributions is a linear continuous operator that commutes with the derivations. Using the standard notations of the theory of distributions ([13], [24]), every convolution operator in $\mathscr{D}'$ is defined by an element $\mu \in \mathscr{E}'$. A particular case of convolution operators are, of course, the partial differential operators with constant coefficients $P(D)$, where P is a complex polynomial in $n+1$ variables, and D stands for the differentiation vector $D = (D_x, D_t) = \left(-i\dfrac{\partial}{\partial x_1}, \ldots, -i\dfrac{\partial}{\partial x_n}, -i\dfrac{\partial}{\partial t}\right)$.

For differential operators, the Cauchy problem can be stated in the following form [12]; [13, Chapter V]:

$$\text{(1.1)} \qquad \begin{cases} \text{Given } f \in \mathscr{D}'(\mathbf{R}^{n+1}) \text{ with } \operatorname{supp} f \subset \mathbf{R}^{n+1}_+, \\ \text{find } g \in \mathscr{D}'(\mathbf{R}^{n+1}) \text{ with } \operatorname{supp} g \subset \mathbf{R}^{n+1}_+, \\ \text{such that } P(D)g = f \text{ in } \mathbf{R}^{n+1}. \end{cases}$$

Hence, the uniqueness problem reduces to study the existence of nontrivial solutions g of the homogeneous equation $P(D)g = 0$, with $\operatorname{supp} g \subset \mathbf{R}^{n+1}_+$.

A classical theorem of Holmgren states that the necessary and sufficient condition

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