QUASICONFORMALLY HOMOGENEOUS CURVES

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A Jordan curve C on the Riemann sphere is called *quasiconformally homogeneous* if for each pair of points P and Q ϵ C there is a quasiconformal map ϕ defined in a neighborhood of C such that $\phi C = C$ and $\phi(P) = Q$. Examples of quasiconformally homogeneous curves are provided by the so-called quasicircles; *i.e.*, quasiconformal images of the unit circle. Other examples are not known, but the question of their existence was raised in [2] by D. K. Blevins and B. P. Palka. It is our purpose to answer this question negatively by proving the following result.

THEOREM 1. *Every quasiconformally homogeneous curve is a quasicircle.*

The proof of Theorem 1 will depend on a local characterization of quasicircles. A *Jordan domain with reference points* is a triple (D, p, p^*), where D is a Jordan domain, $p \in D$, and p^* is a point of the complementary Jordan domain D^*. A *morphism* $(D, p, p^*) \to (D_1, p_1, p_1^*)$ is a quasiconformal map f of the sphere onto itself such that $fD \subset D_1$, $f(p) = p_1$, and $f(p^*) = p_1^*$.

The *dilatation* of (D, p, p^*) is a nonnegative function Δ defined on the boundary $C = \partial D$ as follows. Let U be the open unit disc, and for $P \in C$ denote by $\mathscr{F}(P)$ the family of morphisms $f: (U, 0, \infty) \to (D, p, p^*)$ such that $f(1) = P$. Let K(f) denote the maximal dilatation of f, and define $\Delta(P) = \inf \{K(f): f \in \mathscr{F}(P)\}$. (If $\mathscr{F}(P)$ is empty, then by convention $\Delta(P) = +\infty$.)

LEMMA. *The dilatation is a lower-semicontinuous function which assumes at least one finite value.*

Proof. For $P \in C$ let $m(P) = \liminf_{Q \to P} \Delta(Q)$; we have to show that $\Delta(P) \leq m(P)$. Suppose $m(P) < \infty$, and choose $\varepsilon > 0$. There is a sequence $\{P_i\}$ on C such that $P_i \to P$ and $\Delta(P_i) < m(P) + \varepsilon$ for each i. Choose $f_i \in \mathscr{F}(P_i)$ so that

$$K(f_i) < m(P) + \varepsilon;$$

since $\{f_i\}$ is a normal family [4, Theorem II.5.1], a subsequence converges uniformly to a morphism $f \in \mathscr{F}(P)$. Moreover, $K(f) \leq m(P) + \varepsilon$, and we conclude that $\Delta(P) \leq m(P)$.

To prove the second assertion we may assume that $p = 0$ and $p^* = \infty$, because the dilatation is invariant under Möbius transformations. Choose $P \in C$ so that the absolute value of P is as small as possible. Then the Möbius transformation $z \mapsto Pz$ is in $\mathscr{F}(P)$, hence $\Delta(P) = 1$.

We say that (D, p, p^*) is of *bounded dilatation* if Δ is a bounded function on C. If C is a quasicircle, then (D, p, p^*) and (D^*, p^*, p) are of bounded dilatation, but the converse is less obvious.

THEOREM 2. *Suppose that* (D, p, p^*) *and* (D^*, p^*, p) *are of bounded dilatation. Then the common boundary of* D *and* D^* *is a quasicircle.*

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