A NOTE ON QUATERNIONIC GEOMETRY

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A *quaternionic manifold* is usually thought of as a 4n-dimensional Riemannian manifold (M, g) whose structural group can be reduced to

$$Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1)/\pm 1 .$$

The group $Sp(n) \cdot Sp(1)$ can be considered as the real representation of $Sp(n) \times Sp(1)$ acting on a quaternionic vector space by $(S_n, S_1)(V) = S_n \cdot V \cdot S_1$ [1, 2, 3]. A quaternionic structure on a manifold is equivalent to the existence of a 3-dimensional vector bundle Q of tensors of type (1, 1) with local bases of almost complex structures I, J, K such that $K = IJ = -JI$, and of a metric g being hermitian with respect to each almost complex structure. It is easy to show that Q is an SO(3) bundle.

Picking local almost complex structures I, J, K, with $IJ = K = -JI$, define

$$\Lambda_{AB} C = g(IA, B) IC + g(JA, B) JC + g(KA, B) KC \tag{1}$$

for local vector fields A, B, C. Then Λ is independent of the choice of I, J, K and thus is a tensor field of type (1, 3) on M. The corresponding tensor field of type (0, 4) is denoted by $\underline{\Lambda}$. Let X be a unit vector field, let S_X be the set of unit vector fields Y orthogonal to X such that $\underline{\Lambda}(X, Y, X, Y) = 1$, and let $[S_X]_m$ be the subspace of the tangent space at m generated by S_X. Then Λ and $\underline{\Lambda}$ have the following properties:

(i) $\Lambda_{XY} = -\Lambda_{YX}$;

(ii) $\underline{\Lambda}(X, Y, Z, W) = \underline{\Lambda}(Z, W, X, Y)$;

(iii) $\Lambda^2_{XY} Z = -\underline{\Lambda}(X, Y, X, Y)Z$;

(iv) $\dim[S_X]_m \geq 2$;

(v) $\Lambda_{X\Lambda_{YZ}X} W = g(X, X)\, \Lambda_{YZ} W$;

(vi) $\Lambda_{YZ} = \Lambda_{XY} \Lambda_{XZ}$.

We will show that a tensor field Λ satisfying (i) to (iv) above determines a reduction of the structural group to $Sp(n) \cdot Sp(1)$, and additionally, if Λ satisfies (v) and (vi), Λ can be recovered by equation (1). Thus Λ is analogous to a tensor field J of type (1, 1) such that $J^2 = -1$, whose existence is equivalent to the reduction of the structural group to U(n).

THEOREM 1. (M, g) *is a quaternionic manifold if and only if* M *admits a global tensor field* Λ *of type* (1, 3) *satisfying the axioms* (i) *to* (iv).

Proof. We first show that $Y \in S_X$ implies that $\Lambda_{XY} X = Y$. Since

$$\underline{\Lambda}(X, Y, X, Y) = g(\Lambda_{XY} X, Y) = 1$$

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