INDUCED COBORDISM THEORIES--AN EXAMPLE

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1. INTRODUCTION

The object of this note is to describe a way to construct new cobordism theories, basically by means of one example. First, recall that if M is a manifold, then the total space of the tangent bundle of M is an almost complex manifold, for $\tau(E(\tau_M)) \cong \pi^*\tau_M \oplus \pi^*\tau_M \cong \pi^*\tau_M \otimes_R \mathbb{C}$. If M is also a complex manifold, then τ_M is a complex bundle, so $\tau(E(\tau_M))$ is the complexification of a complex bundle and thus is a quaternionic (or symplectic) vector bundle.

One then introduces the notion of a *weakly weakly almost complex manifold* as a manifold together with a quaternionic structure on the complexification of the normal bundle. In bundle-theoretic terms, there is a fibering $BSp \to BU$ obtained by considering a quaternionic bundle as just a complex bundle, and a map $\otimes\mathbb{C}$: $BO \to BU$ obtained by classifying the complexification of the universal bundle. One may then form the induced fibering

$$\begin{array}{ccc} B & \longrightarrow & BSp \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ BO & \xrightarrow{\ \otimes\mathbb{C}\ } & BU \end{array},$$

and a weakly weakly almost complex manifold is a manifold M together with a chosen equivalence class of liftings of the normal map ν: $M \to BO$ to B. (See Lashof [2] for the precise formalism of manifold with (B, f)-structure.)

Noting that the complexification of a complex bundle is quaternionic shows that the composite $BU \xrightarrow{\pi} BO \xrightarrow{\otimes\mathbb{C}} BU$ lifts to BSp, and hence every weakly almost complex manifold (for which ν lifts to BU) is weakly weakly almost complex.

Following Lashof, one may introduce the cobordism group $\Omega_*^{(B,f)}$ of weakly weakly almost complex manifolds. The main result of this paper is then:

THEOREM. *The forgetful homomorphism* F: $\Omega_*^{(B,f)} \to \mathfrak{N}_*$ *into unoriented cobordism is monic, and one may choose generators* x_i *of* $\mathfrak{N}_* = Z_2[x_i\colon i \neq 2^s - 1]$ *so that the image of* F *is the polynomial subalgebra on the* x_i (i *odd*) *and* x_i^2 (i *even*).

Note. The image of the complex cobordism ring Ω_*^U in $\mathfrak{N}_*$ is the polynomial subalgebra consisting of the squares (Milnor [3]). The odd-dimensional generators needed may be taken to be U/O manifolds in the sense of Smith-Stong [4]; *i.e.*, manifolds for which the complexification of the normal bundle is trivial.

The results will include a general structure theorem (Remark following Lemma 3.1) showing that many theories are 2-torsion, and an analysis of Wall's cobordism theory W_* (section 4) in a form similar to weakly weakly almost complex cobordism.

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