ON THE EULER CHARACTERISTIC OF REAL VARIETIES

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In this paper we give a bound for the Euler characteristic of algebraic varieties in real projective space. This provides a generalization of, and a conceptual framework for, the well-known work of Petrovsky and Oleĭnik [4] which estimates the Euler characteristic of a nonsingular hypersurface by means of a detailed analysis of the critical points of polynomials. Our result applies to any smooth variety. It is, in fact, much easier to state and prove the theorem in its general setting than to notice that it yields the more computationally formulated Petrovsky-Oleĭnik inequality when applied to hypersurfaces.

We obtain this topological information about the algebraic variety $V \subset \mathbb{R}\mathbb{P}^N$ by comparing V with its *complexification* $V_C \subset \mathbb{C}\mathbb{P}^N$. By definition V_C is the complex projective variety of all complex solutions to the polynomials which define V.

Note (*added in revision*). After this manuscript was prepared, the author learned that similar results have been obtained by Kharlamov in [2]. In that paper he proves Theorem 1 and states Proposition 1 as a conjecture.

THEOREM 1. *Let* $V^{2n} \subset \mathbb{R}\mathbb{P}^N$ *be a nonsingular projective real algebraic variety and suppose that* V_C, *the complexification of* V, *is also smooth. Then the Euler characteristic of* V *is bounded by* $|\chi(V)| \leq \dim H^{n,n}(V_C)$.

Remark. We may always assume that V_C is smooth, since a small variation of the defining polynomials eliminates any singularities of V_C while altering V by a diffeomorphism. A smooth V_C admits a Hodge decomposition of its complex cohomology [see 6]

$$H^k(V_C; \mathbb{C}) = \sum_{p+q=k} H^{p,q}(V_C),$$

and the right hand side of the above inequality refers to this decomposition.

Proof of Theorem 1. Complex conjugation $T: \mathbb{C}\mathbb{P}^N \to \mathbb{C}\mathbb{P}^N$,

$$T(z_0, z_1, \cdots, z_N) = (\bar{z}_0, \bar{z}_1, \cdots, \bar{z}_N),$$

restricts to an involution of V_C with V as fixed-point set. Moreover, T is an isometry, and therefore the Euler characteristic of V is equal to the Lefschetz number $L(T, V_C)$. (See [3], p. 76.)

Let J be the almost complex structure on the tangent bundle of V_C. T carries J to the "conjugate" structure $-J$. Acting on forms, then, T^* sends forms of type (p, q) to forms of type (q, p). Therefore, for every p and q, T^* preserves the direct sum $H^{p,q}(V_C) \oplus H^{q,p}(V_C)$. For $p \neq q$, the trace of T^* restricted to this sum is zero. Therefore

$$L(T, V_C) = \sum_{j=0}^{4n} (-1)^j \operatorname{tr}(T^* \mid H^j(V_C)) = \operatorname{tr}(T^* \mid W),$$

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