H^p-DERIVATIVES OF BLASCHKE PRODUCTS

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1. INTRODUCTION

A Blaschke product B is a function defined by a formula

$$(1.1)\qquad B(z, \{a_n\}) = z^m \prod_{a_n \neq 0} \frac{\bar{a}_n}{|a_n|}\left(\frac{a_n - z}{1 - z\bar{a}_n}\right),$$

where $\sum_n (1 - |a_n|) < \infty$, $|a_n| < 1$ for all n, and m is the number of zeros in the sequence $\{a_n\}$. D. Protas [4] has shown that if, in addition,

$$(1.2)\qquad \sum_n (1 - |a_n|)^\alpha = S < \infty$$

for some α in (0, 1/2), then $B' \in H^p$, that is, the integrals

$$(1.3)\qquad \int_0^{2\pi} |B'(re^{i\theta}, \{a_n\})|^p\, d\theta \qquad (0 < r < 1)$$

are bounded when $0 < p \leq 1 - \alpha$.

The work of O. Frostman [2, Theorem IX] shows that (1.2) with $\alpha > 1/2$ does not necessarily imply the boundedness of the integrals (1.3) on $0 < r < 1$ for all positive p.

In this paper we extend the theorem of Protas to higher-order derivatives as follows, and give some relevant counterexamples.

THEOREM 1. *Let* k *be a natural number, and let* $\{a_n\}$ *be a Blaschke sequence such that* (1.2) *holds for some* α *in* $\left(0, \frac{1}{k+1}\right)$. *Then, if* $m = (1 - \alpha)/k$, *there is a constant* C(m, k) *such that*

$$(1.4)\qquad \int_0^{2\pi} \left|\frac{B^{(k)}(re^{i\theta}, \{a_n\})}{B(re^{i\theta}, \{a_n\})}\right|^m d\theta < C(m, k)S \qquad (1/2 < r < 1).$$

Hence $B^{(k)} \in H^p$ *for each* p *in* (0, m].

At each subsequent appearance, the symbol C denotes a positive constant depending either explicitly or implicitly on the parameters indicated. However the value of C may vary from one appearance to the next.

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