PRIMARY SEMIGROUPS

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1. A commutative semigroup S is *primary*, relative to a commutative ring R with identity, in case the semigroup algebra R [S] contains a primary ideal that separates S, in other words, contains no nontrivial difference of two elements of S. Our main results give various properties of primary semigroups in general, and a characterization of finite primary semigroups when R is a suitable field such as $\mathbb{C}$.

Our interest in primary semigroups stems from the fact that each finitely generated commutative semigroup is a subdirect product of finitely many primary semigroups (when R is Noetherian); this is an easy consequence of primary decompositions and the Hilbert basis theorem. It presents primary semigroups as basic building blocks for an important class of semigroups, and it makes their determination of some interest, particularly in the finitely generated case. The choice of R is of secondary importance in this, so long as R yields a wide class of primary semigroups, and there are indications that $\mathbb{C}$ is as good a choice as any. The author is primarily interested in the structure of semigroups, rather than in the interplay between semigroups and rings; maximum generality has not been sought as far as R is concerned.

2. Our main results on primary semigroups are as follows. First, there are three kinds of primary semigroups: relative to each R, a primary semigroup is either a cancellative semigroup, or a nilsemigroup, or what we call a *subelementary* semigroup, that is, the union $S = N \cup C$ of a nilsemigroup N and a cancellative semigroup C, in which N is an ideal and every element of C is cancellative in the whole semigroup. In the last case, C is also primary if S is primary.

All nilsemigroups are primary (relative to each R). For a cancellative semigroup S to be primary, the torsion part of its group of quotients must be locally cyclic (cyclic, if S is finitely generated); the converse holds if S is finitely generated and R is a field K of characteristic 0 that contains all roots of unity (for example $K = \mathbb{C}$). The case where S is subelementary is more difficult. A subelementary semigroup is easily completed into an elementary one (one whose cancellative part is a group), and this does not affect primariness. When $R = K$ as above, and the primary semigroup $S = N \cup G$ is elementary (with G a group), then the torsion part of G is locally cyclic; this is completed by the following necessary condition on the action of G on S (G acts on S by multiplication in S): under the action of any cyclic subgroup of G, all the finite nonzero orbits in S must have the same number of elements. We could not prove the converse holds except when G is cyclic (finite or infinite). However, this suffices to clean the problem in the finite case.

3. These results require a certain amount of preliminary material. The easier basic properties of subelementary semigroups and primary semigroups will be found in Section 1. Section 2 studies prime semigroups, which are defined like primary semigroups but in terms of prime ideals; this is useful for the main results, because when $R = K$ as above, a primary cancellative semigroup is necessarily prime (and conversely). The main results can then be obtained in Section 3.

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