A PROBLEM IN THE CONFORMAL GEOMETRY OF CONVEX SURFACES

Dimitri Koutroufiotis

1. INTRODUCTION

A homothety of E^3 in the sense of elementary Euclidean geometry is a mapping of the form $\vec{x} \to \lambda \vec{x}$; here $\vec{x}$ and λ denote a position vector and a constant. For subsets A and $\bar{A}$ of E^3 we call a mapping $\Phi: A \to \bar{A}$ a *homothety* if it is the restriction to A of a homothety of E^3.

Let S and $\bar{S}$ denote two smooth (C^∞), oriented surfaces in E^3. Suppose that there exists a diffeomorphism Φ between them such that at points and directions corresponding to each other under Φ, the normal curvatures k and $\bar{k}$ satisfy an equation $\bar{k} = ck$, where c is a constant depending on Φ, but neither on position nor on direction. If, in addition, S is not a developable surface and has nowhere-dense umbilics (points where the principal curvatures coincide), then Φ is a homothety up to a Euclidean motion. This local result, which actually holds, *mutatis mutandis,* for hypersurfaces in a space of constant curvature, is a trivial generalization of a theorem due to R. S. Kulkarni [5, p. 95]. It would be of interest to investigate whether a similar statement can be made in case the constant c is replaced by a smooth function ϕ satisfying appropriate assumptions. In this paper we shall show that if S and $\bar{S}$ are ovaloids (that is, compact surfaces in E^3 with positive Gaussian curvature), then the condition $\bar{k} = \phi k$ does indeed imply that S and $\bar{S}$ are essentially homothetic, provided we impose on ϕ a certain mild restriction. Several local and global questions arise naturally; we shall discuss some of them at the end.

We introduce some additional terminology. Let S and $\bar{S}$ be smooth, two-dimensional Riemannian (or pseudo-Riemannian) manifolds. A diffeomorphism $\Phi: S \to \bar{S}$ will be called *conformal* if there exists a smooth function $\phi \neq 0$ on S, the *scale function,* with the property $\langle \Phi_* \alpha, \Phi_* \beta \rangle_{\Phi(P)} = \phi(P) \langle \alpha, \beta \rangle_P$ for all points P in S and all vectors α and β in the tangent space S_P. If (u, v) is a pair of local parameters for S, we may carry it over to $\bar{S}$, using Φ, so that corresponding points are described by the same pair (u, v). We may then say, equivalently, that Φ is conformal if the quadratic forms Λ and $\bar{\Lambda}$ corresponding to the metrics on S and $\bar{S}$ satisfy the condition $\bar{\Lambda} = \phi\Lambda$ in these parameters. In the case of surfaces in E^3, "conformal" with no further specification will always mean conformal with respect to their first fundamental forms.

Received August 12, 1974.
Partially supported by NSF Grant GP-42833.

Michigan Math. J. 22 (1975).