THE DERIVATIVE OF A HOLOMORPHIC FUNCTION IN THE DISK

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1. INTRODUCTION

J. E. McMillan pointed out that the derived function of a univalent holomorphic function in the disk D: $|z| < 1$ is normal in D, in the sense of O. Lehto and K. I. Virtanen [4, p. 47]; for the proof, see [5] or [6]. In this note, we first improve McMillan's result and then investigate relations between local univalency of a holomorphic function f in D and boundary properties of f'. For z, w $\in$ D, write

$$\delta(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|, \qquad \gamma(z, w) = \frac{1}{2}\log\frac{1 + \delta(z, w)}{1 - \delta(z, w)}.$$

If $z \in D$ and $f'(z) \neq 0$, let $\tau(z) \equiv \tau(z, f)$ be the greatest value γ such that f is univalent in the hyperbolic disk $\{\zeta \in D: \gamma(\zeta, z) < \gamma\}$; if $f'(z) = 0$, we set $\tau(z, f) = 0$. Our first result is the following; it has McMillan's theorem as a corollary.

THEOREM 1. *Let* f *be holomorphic in* D, *and suppose that*

$$\inf_{z\in R(r)} \tau(z, f) > 0,$$

where R(r) *is the annulus* $r < |z| < 1$ $(0 < r < 1)$. *Then* f' *is normal in* D.

2. PROOF OF THEOREM 1

Let $\rho(z) \equiv \rho(z, f)$ $(z \in D)$ be the greatest value δ $(0 < \delta < 1)$ such that f is univalent in $\{\zeta \in D: \delta(\zeta, z) < \delta\}$; if $f'(z) = 0$, we set $\rho(z, f) = 0$.

LEMMA 1. *At each point* $z \in D$ *where* $\rho(z, f) > 0$, *we have the inequality*

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1 - |z|^2}\right| \leq \frac{4}{\rho(z, f)(1 - |z|^2)}. \tag{1}$$

Proof. For a fixed δ $(0 < \delta < \rho(z))$, we set $f_z(\xi) \equiv f(\zeta)$, where $(\zeta - z)/(1 - \bar{z}\zeta) = \delta\xi$ for all $\xi \in D$. Then the function

$$f_z(\xi) = f\left(\frac{z + \delta\xi}{1 + \delta\bar{z}\xi}\right)$$

is univalent in $|\xi| < 1$. Applying the Bieberbach inequality $|b_2| \leq 2$ to the coefficient of ξ^2 in the expansion in powers of ξ of the function

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