INNER FUNCTIONS IN THE POLYDISC AND MEASURES ON THE TORUS

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The purpose of this note is to prove a proposition about the structure of R. P. measures, and to use this result in the consideration of some problems about inner functions in the polydisc. In the first section, we give the result about R. P. measures. In the second section, we apply this result to get some information about factoring inner functions, and in the last section we consider some examples.

1. In what follows, C denotes the space of complex numbers, U the open unit disc in C, and T the boundary of U. We consider Borel measures μ carried on $T^N \subseteq C^N$, where N is a positive integer. We fix positive integers k and ℓ such that $k + \ell = N$, and we write

$$C^N = C^k \times C^\ell, \qquad T^N = T^k \times T^\ell, \qquad U^N = U^k \times U^\ell.$$

If $z \in C^N$, we write $z = (\zeta, w)$, where $\zeta \in C^k$ and $w \in C^\ell$. If n is an N-tuple of integers, we write $n = (\alpha, \beta)$, where α is a k-tuple and β is an ℓ-tuple. We use the usual multi-index notation $z^n = z_1^{n_1} \cdots z_N^{n_N}$. If μ is a Borel measure on T, we let

$$\hat{\mu}(n) = \int_{T^N} \bar{z}^n \, d\mu(z),$$

and if $E \subseteq T^N$ is a Borel set, we let μ_E denote the restriction of μ to E. If μ is a Borel measure on T^N, we denote by $\pi\mu$ the measure on T^k such that $(\pi\mu)(E) = \mu(E \times T^\ell)$ for every Borel set $E \subseteq T^k$. We note that if f is a continuous function on T^k, then

$$\int_{T^k} f(\zeta) \, d(\pi\mu)(\zeta) = \int_{T^N} f(\zeta) \, d\mu(\zeta, w).$$

A real Borel measure μ on T^N is said to be an R. P. *measure* if $\hat{\mu}(n) = 0$ whenever not all the n_i have the same sign. The R. P. measures are the measures whose Poisson integrals are the real parts of holomorphic functions (see [3, p. 33]). Finally, m_k denotes Haar measure on T^k, and m_ℓ denotes Haar measure on T^ℓ.

PROPOSITION 1. *Let* μ *be an* R. P. *measure on* T^N, *and let* $E \subseteq T^k$ *be a Borel set such that* $m_k(E) = 0$; *then* $\mu_{E \times T^\ell} = (\pi\mu)_E \times m_\ell$.

Proof. Fix an ℓ-tuple $\beta \neq 0$, say $\beta_i > 0$ for some i. Then, since μ is an R. P. measure, we see that $\hat{\mu}(\alpha, \beta) = 0$ unless $\alpha_j \geq 0$ for $j = 1, \cdots, k$. Now

Received April 13, 1972.

Michigan Math. J. 20 (1973).