QUASI-COMMUTATIVITY OF H-SPACES

F. D. Williams

Two H-space multiplications m and n on a space X are called H-*equivalent* provided there exists an H-map $f: (X, m) \to (X, n)$ that is a homotopy equivalence. H-equivalence is an equivalence relation, and the enumeration of the H-equivalence classes of multiplications is a topic of current interest. A question that arises in this connection is whether each multiplication m on an H-space is H-equivalent to its transpose, the multiplication n given by the relation $n(x, y) = m(y, x)$. This is certainly the case for each group multiplication, for we may choose the homotopy equivalence that takes each element to its inverse. A similar situation exists in loopspace multiplications. In [2], it is shown that every homotopy-Moufang H-space multiplication on a space is H-equivalent to its transpose, and it is suggested there that this relation may hold for all H-space multiplications. Problem 34 of [6] asks whether this is the case for each multiplication on a finite H-complex. In this paper, we develop an obstruction theory for this question, and we produce some counter-examples. We exhibit a multiplication on a generalized Eilenberg-MacLane space that is not H-equivalent to its transpose, and we demonstrate the existence of a multiplication on 3-dimensional real projective space that has the same property.

PROPOSITION A. *Let* $E = K(Z_2, 1) \times K(Z_2, 4)$. *The multiplication* m *on* E *characterized by making the composition*

$$K(Z_2, 1)^2 \xrightarrow{i_1^2} E^2 \xrightarrow{m} E \xrightarrow{P_2} K(Z_2, 4)$$

correspond to $x \otimes x^3 \in H^4(K(Z_2, 1) \wedge K(Z_2, 1); Z_2)$ *is not* H-*equivalent to its transpose.*

We could verify this proposition directly by computing the elements involved in the appropriate cohomology groups. Instead, we shall give an alternate verification, as an application of the general theory.

PROPOSITION B. *There exists an* H-*space multiplication on the real projective* 3-*space* P_3 *that is not* H-*equivalent to its transpose.*

We now develop the formalism necessary to handle these propositions. Corresponding to a pointed space X, we recall the space FX of unbased paths of varying lengths on X. The elements of FX are maps $\lambda: [0, \infty] \to X$ that are constant in some neighborhood of ∞. The basepoint of FX is the constant path at the basepoint of X. We shall consider the projections π_0 and π_∞ of FX to X given by evaluation at 0 and ∞, respectively. We make FX into a groupoid as follows. Let λ_1 and λ_2 in FX be such that $\pi_\infty(\lambda_1) = \pi_0(\lambda_2)$. Then define their *sum* $\lambda_1 + \lambda_2$ by the formula

$$(\lambda_1 + \lambda_2)[t] = \begin{cases} \lambda_1(t) & (0 \le t \le r_1), \\ \lambda_2(t - r_1) & (r_1 \le t \le \infty), \end{cases}$$

where

Received December 8, 1971.
This research was supported in part by NSF Grant GP-17757.

Michigan Math. J. 19 (1972).