ON EXTREME DOUBLY STOCHASTIC MEASURES

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1. INTRODUCTION

This paper deals with the problem of extremality in the convex set of doubly stochastic measures defined on the unit square $X \times X$ with the Lebesgue structure. By m and m^2, we shall denote one- and two-dimensional Lebesgue measure. A doubly stochastic measure is a positive Borel measure μ defined on $X \times X$ such that

$$\mu(A \times X) = \mu(X \times A) = m(A)$$

for every measurable A.

The major results of this paper are called theorems, and the technical results are called propositions. We use the Douglas-Lindenstrauss Theorem:

Let μ be a doubly stochastic measure. The measure μ is extreme if and only if the subspace consisting of all functions of the form $h(x, y) = f(x) + g(y)$ $(f, g \in L_1(m))$ *is norm-dense in* $L_1(\mu)$.

R. G. Douglas [1] and Joram Lindenstrauss [4] discovered this theorem independently. It is the only known characterization of the extreme points of the set of doubly stochastic measures.

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2. RESULTS

Every example, known to the authors, of an extreme point in the set of doubly stochastic measures has the mass concentrated along line segments. In fact, there exist examples in which the mass seems to be concentrated on points.

Definition 1. The point (x, y) is a *point of density of the set* E, *relative to the measure* μ, if

$$\mu[([x - h, x + h] \times [y - h, y + h]) \cap E] > 0$$

for every $h > 0$.

The set of density points of E that are in E forms a closed subset of E in its relative topology. This collection is essentially the support of μ restricted to E. However, there may be points of density for E that are not in E. Proposition 1 and its corollaries are addressed to this larger collection of points.

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