A NOTE ON INVARIANT SUBSPACES

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P. R. Halmos and L. J. Wallen have asked whether on Hilbert space, there exists an algebraic linear transformation (not necessarily continuous) having no proper closed invariant subspace. In this note, we show that such transformations do exist. We also show, using a result of H. H. Schaefer [5], that every continuous linear transformation on the Fréchet space (s) of all sequences has a proper closed invariant subspace.

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THEOREM 1. *Let* H *denote a separable, infinite-dimensional, complex Hilbert space. Then there exists an algebraic linear transformation* T *of* H *into itself having no proper closed invariant subspace.*

Proof. Let c denote the cardinal number of the continuum, and let ω_c denote the first transfinite ordinal number for which the set of all smaller ordinal numbers has cardinality c. The set of all infinite-dimensional proper closed subspaces of H has cardinality c; well-order this set in a minimal well-ordering. That is, assign to each ordinal number α $(1 \leq \alpha < \omega_c)$ an infinite-dimensional proper closed subspace M_α such that each subspace occurs exactly once.

Using transfinite induction, we shall assign two vectors f_α, g_α to each ordinal number $\alpha < \omega_c$ such that

i) $f_\alpha \in M_\alpha$,

ii) $g_\alpha \notin M_\alpha$,

iii) the set of all vectors f and g is linearly independent.

Choose f_1 and g_1 to satisfy i), ii), iii) above for $\alpha = 1$. Now assume that $\alpha < \omega_c$ is given and that f_β, g_β have been chosen for all $\beta < \alpha$. Then the vector subspace V_α determined by all these f_β and g_β cannot contain M_α, since the cardinality of a Hamel basis for M_α is c. For f_α, select some vector that is in M_α but not in V_α. The vector subspace W_α determined by V_α and f_α is not all of H, since it has a Hamel basis of cardinality less than c; therefore it cannot contain any nonvoid open subset of H. Hence the set-theoretic union of M_α and W_α is not all of H. For g_α, choose some vector not in this union. Then the set of all vectors f_β, g_β $(\beta \leq \alpha)$ is linearly independent, and the induction is complete.

Extend the set of all f_α, g_α $(\alpha < \omega_c)$ to a Hamel basis for H by the adjunction of a set $\{h_\gamma\}$. Define the linear transformation T by requiring that

$$Tf_\alpha = g_\alpha, \quad Tg_\alpha = f_{\alpha+1} \qquad (\alpha < \omega_c).$$

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