ON ADDITIVE FUNCTIONS

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Introduction. A number-theoretic function f is said to be *additive* if $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$; we denote the class of such functions by $\mathscr{A}$. Because of the special nature of the subclass $\mathscr{B}$ of functions of the form $f(n) = c \log n$, it is of interest to find conditions on functions f in $\mathscr{A}$ under which f is also in $\mathscr{B}$.

The first investigation in this direction was made by P. Erdös [1], who proved that if $f \in \mathscr{A}$ and $f(n+1) - f(n) \geq 0$ for each natural number n, then $f \in \mathscr{B}$. Erdös conjectured that the same conclusion holds if the monotonicity condition is relaxed to the requirement that $f(n+1) - f(n) \geq 0$ for almost all n, and this conjecture was subsequently proved by I. Kátai [2]. Erdös also proved that if $f \in \mathscr{A}$ and $\lim_{n\to\infty} [f(n+1) - f(n)] = 0$, then $f \in \mathscr{B}$, and he conjectured that the condition on $f(n+1) - f(n)$ can be replaced by the condition

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} |f(n+1) - f(n)| = 0 .$$

The last conjecture was recently established by Kátai (proof to appear). E. Wirsing subsequently found an elegant proof of this result, and since the proof of Theorem 1 is based on some of the ideas in his proof, we shall give an outline of his method, at the end of Lemma 3.

Finally, we mention a long-standing conjecture of Erdös, recently proved by Wirsing [3]:

THEOREM (Wirsing). *Suppose that* $f \in \mathscr{A}$ *and that the set of differences* $f(n+1) - f(n)$ *is bounded. Then* $f(n) = c \log n + g(n)$, *where* g *is a bounded, additive function.*

Some time ago, I conjectured that the following is true:

CONJECTURE. *If* $f \in \mathscr{A}$ *and*

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\leq x} |f(n+1) - f(n)| = 0 , \tag{1}$$

then $f \in \mathscr{B}$.

The conjecture is still open; but in this paper I prove the following weaker version of it.

THEOREM 1. *Let* $f \in \mathscr{A}$, *and let* f *satisfy condition* (1) *and*

$$f(n) = O(\log n) . \tag{2}$$

Then $f \in \mathscr{B}$.

I shall also prove the following result.

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