VON NEUMANN ALGEBRAS WITH A SINGLE GENERATOR

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A *von Neumann algebra* is a weakly closed, self-adjoint algebra of operators on a (complex) Hilbert space with the property that the identity operator on the Hilbert space belongs to the algebra. If $\{A_1, A_2, \cdots\}$ is a finite or countably infinite collection of operators acting on the Hilbert space $\mathfrak{H}$, then the *von Neumann algebra generated by the collection* $\{A_1, A_2, \cdots\}$ is by definition the smallest von Neumann algebra that contains each operator A_i; we denote this algebra by $\mathscr{R}(A_1, A_2, \cdots)$. It is known [1, p. 33] that if $\mathfrak{H}$ is separable, then every von Neumann algebra $\mathscr{A}$ acting on $\mathfrak{H}$ may be written as $\mathscr{A} = \mathscr{R}(A_1, A_2, \cdots)$ for some countable family $\{A_1, A_2, \cdots\}$ of operators in $\mathscr{A}$. A von Neumann algebra $\mathscr{A}$ is said to have a *single generator* if there exists an operator A in $\mathscr{A}$ such that $\mathscr{A} = \mathscr{R}(A)$. (It is easy to show that $\mathscr{A} = \mathscr{R}(A)$ if and only if $\mathscr{A}$ consists of the weak closure of the set of all polynomials $p(A, A^*)$ in A and A^*.)

Problem. Does every von Neumann algebra $\mathscr{A}$ acting on a separable Hilbert space have a single generator?

This problem has been before us for some time. The first result bearing on it is the theorem of von Neumann [4] that if $\mathscr{A}$ is abelian, then $\mathscr{A}$ has a single Hermitian generator. Further progress was made by Pearcy, who showed in [5] that $\mathscr{A}$ has a single generator if it is of type I, and who introduced in [6] a certain matricial technique that has turned out to be useful in subsequent work on this problem. Next, Suzuki and Saitô proved in [10] that if $\mathscr{A}$ is hyperfinite, then $\mathscr{A}$ has a single generator (see [3, footnote 68]). Finally, Wogen [11] recently extended certain important results of Saitô [9], and he used the extensions to prove that if $\mathscr{A}$ is properly infinite (that is, if $\mathscr{A}$ contains no nonzero finite central projection), then $\mathscr{A}$ has a single generator.

In most of these papers, von Neumann algebras $\mathscr{A}$ having the property

(T) $\mathscr{A}$ is (algebraically) * - isomorphic to the von Neumann algebra $M_2(\mathscr{A})$ of all 2×2 matrices over $\mathscr{A}$

play a central role.

The problem of identifying the von Neumann algebras with property (T) is difficult. In particular, it is known [3] that certain von Neumann algebras of type II_1 have property (T), but it is not known whether *every* von Neumann algebra of type II_1 has property (T).

The purpose of this note is to prove the following two theorems.

THEOREM 1. *Suppose that $\mathscr{A}$ is a von Neumann algebra of type II_1 acting on a separable, infinite-dimensional Hilbert space $\mathfrak{H}$. Suppose also that every von Neumann subalgebra of $\mathscr{A}$ that is of type II_1 (acting perhaps on a smaller space) has property* (T). *Then $\mathscr{A}$ has a single generator.*

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