ASSOCIATED FIBRE SPACES

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From the point of view of homotopy theory, this paper looks at transformation groups and the association between G-bundles and principal G-bundles. It makes fundamental extensions to a theory of "transformation monoids," and it discusses a correspondence between quasifibrations and associated principal quasifibrations.

1. INTRODUCTION

In the theory of transformation groups, orbit spaces play an important role. It is particularly helpful if the map $X \to X/G$ is a fibre bundle, in fact, a principal G-bundle. In the theory of fibre bundles, the correspondence between G-bundles with fibre F and associated principal G-bundles is of crucial importance. If $p: E \to B$ is a (right) principal G-bundle and G is represented as a transformation group on F, the associated G-bundle $q: E \times_G F \to B$ with fibre F is defined by $E \times_G F = E \times F/G$, where G acts via the diagonal action $g(e, f) = (eg^{-1}, gf)$. If G is a transformation group on X and $X \to X/G$ is not a bundle, we can study the Borel bundle $\mathcal{E}_G \times X \to \mathcal{E}_G \times_G X = X_G$ [2], where $\mathcal{E}_G \to B_G$ is a universal principal G-bundle. [We use the notation $\mathcal{E}_G$, in contrast to [2] and [4], so that E_G can refer unambiguously to the above construction with $X = E$. There is no space $\mathcal{E}$ in this paper.] The total space $\tilde{X} = \mathcal{E}_G \times X$ has the same weak homotopy type as X, and if $X \to X/G$ were a principal G-bundle, X_G would have the same weak homotopy type as X/G.

Once we have adopted the point of view of weak homotopy type, it is natural to consider fibre spaces and even quasifibrations [5] instead of bundles, and weak homotopy equivalences instead of homeomorphisms. Since weak homotopy equivalences do not have precise inverses, it is appropriate to look at monoids of weak homotopy equivalences rather than groups thereof. This paper is an initial contribution to the theory of *transformation monoids* with particular emphasis on the role of (quasi-) fibrations.

We begin by fixing some elementary notation and terminology.

Definition 1.1. Let M be a topological monoid. A (left) M-*space* $(X; \mu)$ consists of a space X and a map $\mu: M \times X \to X$ such that (with $\mu(m, x) = mx$)

$$1)\ m(nx) = (mn)x, \qquad 2)\ ex = x.$$

If $\mu(m,\ ): X \to X$ is a weak homotopy equivalence of X for each $m \in M$, we say $(X; \mu)$ is a *representation of* M *by weak homotopy equivalences of* X or simply a *weak representation of* M. The adjoint to μ is a function $M \to X^X$, and it is a homomorphism. For a *right* M-space, the adjoint is an antihomomorphism. We then speak of a *weak antirepresentation of* M.

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