WILD POINTS OF CELLULAR SUBSETS OF SPHERES IN S^3, II

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The set of wild points of a cellular finite graph on a 2-sphere in S^3 is either empty or degenerate, or it contains an arc [6, Theorem 1]. Furthermore, such a finite graph cannot contain two isolated wild points [6, Theorem 2]. The purpose of this note is to indicate how we can use a theorem recently proved by D. R. McMillan, Jr. [9, Theorem 1], together with the results of [6], to obtain similar results for arbitrary cellular subsets of 2-spheres in S^3.

A key to the proofs in [6] is the result by Burgess [4] that a 2-sphere S in S^3 has at most two wild points if each component of S^3 - S is an open 3-cell and S is locally tame modulo a 0-dimensional set.

A subset X of S^3 is said to be cellular if there exists a sequence of 3-cells $C_1, C_2, \cdots$ such that $X = \bigcap_1^\infty C_i$ and $C_{i+1} \subset \text{Int } C_i$ for each i. (For other definitions, consult the references.)

THEOREM 1. *A cellular subset of a 2-sphere* S *in* S^3 *cannot contain two isolated wild points of* S.

Proof. Suppose X is a cellular subset of S such that X contains two isolated wild points p and q of S. Some arc A in S contains points p and q such that A is locally tame modulo $\{p, q\}$ (see [7] or [8]), and some disk D containing $X \cup A$ is locally polyhedral modulo $X \cup A$ [2]. Since A is locally tame modulo two points of X, we see that D is locally tame modulo X [5]; hence D is cellular. According to McMillan [9], this implies that A is cellular. It follows that A is locally tame either at p or at q [6], say at p. Since p is an isolated wild point of S and p lies in a tame arc in S, we have a contradiction [5].

Note. There are examples of cellular arcs that lie on a 2-sphere and contain exactly two wild points [1] (in fact, infinitely many wild points [3]) of the sphere.

THEOREM 2. *If* X *is a cellular subset of a* 2-*sphere* S *in* S^3 *such that* S *is locally tame* modulo X *and the set* W *of wild points of* S *is* 0-*dimensional, then* W *contains at most one point.*

Proof. There exists an arc A in S containing W [10], and there exists a disk D in S such that $A \cup X \subset D$. Since D is locally tame modulo X, it follows that D is cellular; hence A is also cellular [9]. Therefore A has at most one wild point [6], and it follows from [5] that W contains at most one point.

THEOREM 3. *If* X *is a cellular subset of a* 2-*sphere* S *in* S^3 *and* S *is locally tame* modulo X, *and if* W *is the set of wild points of* S, *then either* W *is empty,* W *is degenerate, or* W *contains a nondegenerate continuum.*

Furthermore, if U *is an open subset of* S *such that* $U \cap W$ *is* 0-*dimensional, then* $W \cap U$ *contains at most one point.*

Proof. Suppose every continuum in W is degenerate. Then W is totally disconnected — hence W is 0-dimensional. It follows from Theorem 2 that W contains at most one point.

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