# COMPLETE DISTRIBUTIVITY IN CERTAIN INFINITE PERMUTATION GROUPS

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## 1. INTRODUCTION

An $\ell$-group G is said to be *completely distributive* if the order of constructing infinite joins and intersections may be interchanged. In 1939, Lorenzen [7] proved that an abelian $\ell$-group can be embedded in a large cardinal product of totally ordered groups. In 1963, Conrad, Harvey, and Holland [4] showed that an abelian $\ell$-group can be realized as an $\ell$-subgroup of an $\ell$-group of real-valued functions. Both of these embedding theorems present an abelian $\ell$-group as an $\ell$-subgroup of a completely distributive $\ell$-group. In 1963, Holland [6] proved that any $\ell$-group can be embedded in the group of order-preserving permutations of some totally ordered set. The main purpose of this note is to show that the Holland embedding realizes any $\ell$-group as an $\ell$-subgroup of a completely distributive $\ell$-group.

Section 3 is devoted to proving that the group P(L) of order-preserving permutations of a totally ordered set L is a completely distributive $\ell$-group. It follows as a corollary that the ideal radical of P(L) is trivial. In Section 4 it is shown that the isotropy subgroups of P(L) are closed convex $\ell$-subgroups. In Section 5 we answer a question raised by Conrad [3], by giving an example of an $\ell$-group that has a trivial ideal radical and yet fails to be completely distributive.

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## 2. NOTATION AND TERMINOLOGY

For standard results and definitions concerning $\ell$-groups, the reader is referred to [1] and [5]. If G is an $\ell$-group, $G^+ = \{x \in G \mid x \geq 1\}$ is called the *positive cone* of G. An $\ell$-group G is said to be *completely distributive* if the relation

$$\bigwedge_{i \in I} \left( \bigvee_{j \in J} g_{ij} \right) = \bigvee_{f \in J^I} \left( \bigwedge_{i \in I} g_{if(i)} \right)$$

holds whenever $\{g_{ij} \mid i \in I,\ j \in J\}$ is a subset of G for which all the indicated joins and intersections exist.

If L is a totally ordered set, P(L) denotes the collection of order-preserving permutations of L. P(L) is a group under the operation of composition of functions, and it is an $\ell$-group with respect to the partial order defined by the rule

$$f \geq g \text{ if and only if } f(x) \geq g(x) \text{ for each } x \in L.$$

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